Some Graded Lie Algebra Structures Associated with Lie Algebras and Lie Algebroids

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Abstract

The main objects of this thesis are graded Lie algebras associated with a Lie algebra or a Lie algebroid such as the Frölicher-Nijenhuis algebra, the Kodaira-Spencer algebra and the newly constructed Gelfand-Dorfman algebra and generalized Nijenhuis-Richardson algebra. Main results are summarized as follows: We introduce a derived bracket which contains the Frölicher-Nijenhuis bracket as a special case and prove an interesting formula for this derived bracket. We develop a rigorous mechanism for the Kodaira-Spencer algebra, reveal its relation with R-matrices in the sense of M. A. Semenov-Tian-Shansky and construct from it a new example of the knit product structures of graded Lie algebras. For a given Lie algebra, we construct a new graded Lie algebra called the Gelfand-Dorfman algebra which provides for r-matrices a graded Lie algebra background and includes the well-known Schouten-Nijenhuis algebra of the Lie algebra as a subalgebra. We establish an anti-homomorphism from this graded Lie algebra to the Nijenhuis-Richardson algebra of the dual space of the Lie algebra, which sheds new light on our understanding of Drinfeld's construction of Lie algebra structures on the dual space with r-matrices. In addition, we generalize the Nijenhuis-Richardson algebra from the vector space case to the vector bundle case so that Lie algebroids on a vector bundle are defined by this generalized Nijenhuis-Richardson algebra. We prove that this generalized Nijenhuis-Richardson algebra is isomorphic to both the linear Schouten-Nijenhuis algebra on the dual bundle of the vector bundle and the derivation algebra associated with the exterior algebra bundle of this dual bundle. A concept of a $2n$-ary Lie algebroid is proposed as an application of these isomorphisms.

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Contents

Chapter 1

Introduction

Various constructions and algebra structures can be described in terms of degree 1, bracket-square 0 elements of graded Lie algebras. Such descriptions usually provide new perspectives when we are dealing with some problems associated with these constructions and structures. This is clear in algebraic deformation theory ([GS], see also [LMS]). The Gerstanhaber algebra and the Nijenhuis-Richardson algebra are powerful tools in the study of deformations of associative and Lie algebras respectively ([G1] and [NR2]). It is also clear in differential geometry. Examples here includes the characterization of Poisson structures on a manifold through the Schouten-Nijenhuis algebra over the manifold ([V1]) and the Newlander-Nirenberg theorem in terms of the Frölicher-Nijenhuis algebra over a manifold ([NN] and [FN1,2]). We will construct in this thesis two new graded Lie algebra structures which are called the Gelfand-Dorfman algebra for a Lie algebra and the generalized Nijenhuis-Richardson algebra over a vector bundle and provides some new insights into the well-known Frölicher-Nijenhuis algebra and the Kodaira-Spencer algebra. These graded Lie algebras describe such important mathematical objects as r-matrices, Lie algebroids, R-matrices and Nijenhuis operators in the above-mentioned manner.

1.1 Main Results

We list our main results in order of their appearance in the body of this thesis.

1.1.1 The Frölicher-Nijenhuis Algebra

The Frölicher-Nijenhuis algebra on $Alt(V, V)$ for a Lie algebra V was studied in [N2]. Its degree 1, bracket-square 0 elements are sometimes called Nijenhuis operators. A Nijenhuis operator induces a second Lie algebra structure on V and this new Lie algebra structure plays an important role in the bihamiltonain method of studying completely integrable Hamiltonian systcms ([D] and [K-SM], see also [MM]).

In Chapter 4, we will introduce a bracket on $Alt(V, V)$ which is derived from the Nijenhuis-Richardson bracket on $Alt(V, V)[1]$ (a graded vector space obtained by shifting $Alt(V, V)$ down by 1 degree) and which contains the Frölicher-Nijenhuis bracket as a special case. We particularly focus on a formula associated with this derived bracket (Theorem 4.5). Such a formula is established in [N2] for the Frölicher-Nijenhuis bracket to express the Frölicher-Nijenhuis bracket for the new Lie algebra on V induced by a Nijenhuis operator in terms of the Frölicher-Nijenhuis bracket for the original Lie algebra on V. While it is not diffcult to realize that Nijenhuis' formula holds for our more general derived bracket, the proof of this formula in our thesis is new.

1.1.2 The Kodaira-Spencer Algebra

In Chapter 5, we establish the Kodaira-Spencer algebra on $Alt(V, V)$ for a Lie algebra V. It provides a graded Lie algebra description of both the classical and the modified classical Yang-Baxter equations associated with the Lie algebra V in the sense of Semenov-Tian-Shansky ([STS]). Some interesting results of the Kodaira-Spencer algebra follow from our approach to its constructon. For example, we easily have that an interesting operator Θ is a homomorphism from the Kodaira-Spencer algebra to the Nijenhuis-Richardson algebra of the underlying vector space of the Lie algebra V (see (5.4)). The fact that R-matrices, as solutions to Yang-Baxter equations, define new Lie algebra structures on V becomes a direct consequence of this homomorphism.

The Kodaira-Spencer algebra was originally defined on the graded vector space of vector-valued differential forms on a manifold ([KS] and [BM]). To my knowledge, the version we consider in this thesis has not been studied before.

1.1.3 Knit Product Structures

A knit product is a graded Lie algebra structure on the direct sum of two graded Lie algebras when they have mutual representations on each other satisfying certain conditions (§2.1.5). In Chapter 4, we have a more clear (compared with [N2]) and more straight (compared with [Mi]) exposition of the knit product of the Nijenhuis-Richardson algebra $Alt(V, V)[1]$ ([NR2]) and the Frölicher-Nijenhuis algebra $Alt(V, V)$ (Theorem 4.4). In addition, we show in Chapter 5 there exists a knit product structure between the Nijenhuis-Richardson algebra and the Kodaira-Spencer algebra (Theorem 5.6). As far as I know, this is only the second example of a knit product of graded Lie algebras.

We point out constructions similar to the knit product have been studied for some other algebra structures in mathematics. For exmple, Majid considered the Lie algebra case and coined the name a matched pair ([M]) for two Lie algebras from the direct sum of which a new Lie algebra can be constructed. Mokri studied a matched pair of Lie algebroids ([Mo]). The newly constructed structure is called a twilled extension for Lie algebras by Kosmann-Schwarzbach and Magri ([K-SM]) and for Lie-Reinhart algebras by Huebschmann ([H]). The name of a knit product for graded Lie algebras is given by Michor ([Mi]).

1.1.4 The Gelfand-Dorfman Algebra

The first new graded Lie algebra we construct in this thesis is the Gelfand-Dorfman algebra $\bigwedge V \bigotimes V$ for a Lie algebra V. Its degree 1, bracket-square 0 elements are general (not necessarily anti-symmetric) r-matrices of the Lie algebra V ([Dr1,2]).

In Chapter 6, besides the construction of the Gelfand-Dorfman algebra (Theorem 6.1), we establish two results. First, we show that the Gelfand-Dorfman algebra contains a subalgebra isomorphic to the Schouten-Nijenhuis algebra (Theorem 6.7). This is a natural result since anti-symmetric r-matrices are degree 1, bracket square 0 elements of the Schouten-Nijenhuis algebra. Second, we establish an anti-homomorphism from the Gelfand-Dorfman algebra to the Nijenhuis-Richardson algebra $Alt(V^*, V^*)[1]$ for the vector space V^* (Theorem 6.9). This anti-homomorphism generalizes a construction of Drinfeld in the Poisson-Lie group theory (see Proposition 6.8).

1.1.5 The Generalized Nijenhuis-Richardson Algebra

To describe the generalized Nijenhuis-Richardson algebra, it is convenient to recall the notion of a Lie algebroid ([Ma1,2]) first. A Lie algebroid over a smooth manifold M is a vector bundle A over M together with a Lie algebra structure on the space $\Gamma(A)$ of smooth sections of A and a bundle map $\rho : A \longrightarrow TM$ such that ρ defines a Lie algebra homomorphism from $\Gamma(A)$ to $\mathbf{X}(M)$, the Lie algebra of vector fields over M, and there holds for $f \in C^\infty(M)$ and $\xi_1, \xi_2 \in \Gamma(A)$, the following derivation law,

$$[f\xi_1, \xi_2] = f[\xi_1, \xi_2] - \rho(\xi_2)f \cdot \xi_1.$$

A Lie algebroid is a generalization of a Lie algebra. The natural question is : what is the graded Lie algebra on a vector bundle which defines Lie algebroid structures? In Chapter 7, we construct such a graded Lie algebra $LR(A)$ for a vector bundle A through a generalization of the Nijenhuis- Richardson algebra from the vector space case to the vector bundle one (Theorem 7.3).

It is known that Lie algebroids on a vector bundle A are in one-one correspondence with linear Poisson structures on its dual bundle A^* ([C] and [CDW], see also [W1]). This is sometimes called the generalized Lie-Poisson construction. In Chapter 7, we point out that linear polyvector fields on a vector bundle constitute a subalgebra of the Schouten-Nijenhuis algebra over the bundle (considered as a manifold), which will be called the linear Schouten-Nijenhuis algebra over the bundle, and prove that the generalized Nijenhuis-Richardson algebra for a vector bundle is isomorphic to the linear Schouten-Nijenhuis algebra over its dual bundle (Theorem 7.10). Since the degree 1, bracket-square 0 elements of the linear Schouten-Nijenhuis algebra define linear Poisson structures, our result extends the generalized Lie-Poisson construction. In the course of developing Theorem 7.10, we also give a different proof of the following result: the Schouten-Nijenhuis algebra over a manifold N is a subalgebra of the Nijenhuis-Richardson algebra for the vector space $C^\infty(N)$ ([CKMV]).

We also extend another correspondence in the Lie algebroid theory, the correspondence between Lie algebroids on a vector bundle A and 1-differentials of sections of the exterior algebra bundle of its dual bundle A^* ([K-SM] and [X]). We establish an isomorphism between the generalized Nijenhuis-Richardson algebra for A and the derivation algebra of the above-mentioned exterior algebra of sections (Theorem 7.11). This isomorphism also generalizes the classical work of Frölicher and Nijenhuis on the characterization of the derivation ring of differential forms on a smooth manifold ([FN1]).

1.2 Techniques behind the Results

We briefly discuss here some ideas we use to develop our main results.

The semidirect product (Theorem 3.2) of the Nijenhuis-Richardson algebra $Alt(V, V)[1]$ for a vector space V and the Lie induced algebra $Alt(V, W)$ associated with V and a Lie algebra W ([NR3]) plays an important role in developing some of our main results. The generalized Nijenhuis-Richardson algebra $LR(A)$ is constructed as a subalgebra of this semidirect product with $V = \Gamma(A)$ and $W = \mathbf{X}(M)$. It is through a special case of this semidirect product ($V = W$) that we get an effective way to attain the two knit products in §1.1.3. This special case of the semidirect product was already used in [N2]. However, as far as I know, the general construction of the semidirect product is considered in this thesis for the first time.

In Chapter 3, we introduce an operator Θ which is "almost" the difference of two coboundary operators δ and D in the Lie algebra cohomology theory (see(3.8)). The operator δ is for the adjoint representation and D is for the trivial representation on the Lie algebra itself. However, it displays a fundamentally different property (Proposition 3.8) compared with the property of δ and D (Lemma 3.7). Though, we find that with the place of the operator δ in [N2] taken by this operator Θ Nijenhuis' idea there still works well with necessary modifications. This leads to the Kodaira-Spencer algebra $Alt(V, V)$ and to some of its remarkable properties.

Mainly for readers' convenience of comparing the Kodaira-Spencer algebra with the Frölicher-Nijenhuis algebra, we include in Chapter 4 the mechanism used by Nijenhuis in constructing the latter algebra ([N2]). Nijenhuis showed that the Frölicher-Nijenhuis bracket is essentially a measure of the deviation of δ from being a derivation of the composition product on $Alt(V, V)$. Graded Lie brackets of this kind turn out to be fairly commmon in mathematics and mathematical physics. An example is the Batalin-Vilkovisky algebra (see [K-S4] and references therein). We do not know at this moment whether brackets of the Kodaira-Spencer form as in (5.3) will find applications in physics.

1.3 The Structure of this Thesis

In Chapter 2, we introduce standard definitions and constructions in graded Lie algebra theory. Then the shuffle algebra is introduced together with multi-shuffles as a tool in dealing with some complicated computations in this thesis. In the third part of Chapter 2, we list some classical examples of a graded Lie algebra, including two versions of the Schouten-Nijenhuis algebra, the Nijenhuis-Richardson algebra and the Lie induced algebra.

The first section of the Chapter 3 constructs the semidirect product of the Nijenhuis-Richardson algebra and the Lie induced algebra. In the second part, we introduce the operators δ and Θ and study their interaction with the cup algebra $Alt(V, V)$ which is the special case of Lie induced algebra $Alt(V, W)$.

Chapter 4, 5 and 6 study the Frölicher-Nijenhuis algebra, the Kodaira-Spencer algebra and the Gelfand-Dorfman algebra respectively.

In Chapter 7, we first construct the Nijenhuis-Richardson algebra, then prove two isomorphism theorems mentioned in §1.1.5. Finally, through the introduction of $2n$-ary Lie algebroids, we illustrate, in a more general setting, the implications of these isomorphisms for the Lie algebroid theory.

1.4 Cast of Characters

For convenience of the reader, we summarize in the following table graded Lie algebras which appear in this thesis.

Graded Lie Algebra	**Graded Vector Space**	**Graded Lie Bracket**	**Meaning of Degree 1, Bracket-Square 0 Element**
Schouten-Nijenhuis Algebra(1)	$\bigwedge V[1]$	(2.10)	anti-symmetric r-matrix
Schouten-Nijenhuis Algebra(2)	$\bigwedge TN$	(2.10) and others	Poisson structure
Nijenhuis-Richardson Algebra	$Alt(V,V)[1]$ $\bigwedge V^* \otimes V[1]$	(2.12) and (2.11) (2.12) and (2.14)	Lie algebra structure
Lie Induced Algebra	$Alt(V,W)$ $\bigwedge V^* \otimes W$	(2.16) (2.17)	
Frölicher-Nijenhuis Algebra	$Alt(V,V)$ $\bigwedge V^* \otimes V$	(4.3) or (4.12) (4.13)	Nijenhuis operator
Kodaira-Spencer Algebra	$Alt(V,V)$ $\bigwedge V^* \otimes V$	(5.3) 0r (5.5) (5.6)	R-matrix
Gelfand-Dorfman Algebra	$Alt(V^*,V)$ $\bigwedge V \otimes V$	(6.9) (6.3)	r-matrix
Generalized Nijenhuis-Richardson Algebra	$LR(A)$	cf.(3.4)	Lie algebroid

Chapter 2

Preliminaries

Without specification, all objects in this thesis are over real numbers R , and all vector spaces and Lie algebras with the only exception of those in the last chapter are finite-dimensional.

2.1 Graded Lie Algebras

Our survey of graded Lie algebras in this section is mainly based on [NR1] with the exception of the knit product which is adapted from [Mi].

2.1.1 Basic Definitions

A graded vector space is a vector space B together with a family $\{B^k\}_{k\in Z}$ of subspaces of B, indexed by Z, such that B is the direct sum of the family $\{B^k\}_{k\in Z}$. The elements of B^k are called homogeneous of degree k. Graded subspaces are defined in the obvious way. If $B = \oplus_{k\in Z}B^k$ and $C = \oplus_{k\in Z}C^k$ are two graded vector spaces, then their direct sum is a new graded vector space $B \oplus C = \oplus_{k\in Z}(B^k \oplus C^k)$.

A linear map l of a graded vector space $B = \oplus_{k\in Z}B^k$ into a graded vector space $C = \oplus_{k\in Z}C^k$ is homogeneous of degree m if for every $k \in Z$, $l(B^k) \subset C^{k+m}$. In particular, shift operators $[m]$ are of homogeneous degree m. They act on elements as the identity but shift their degrees down by m. In other words, given a graded vector space $B = \oplus_{k\in Z}B^k$, $B[m]$ is a new graded vector space with $B[m]^k = B^{k+m}$, $k \in Z$.

A graded algebra is a graded vector space $B = \oplus_{k\in Z}B^k$ which is given an algebra structure compatible with its graded structures, i.e., a bilinear map $(b_1, b_2) \to b_1b_2$ of

$B \times B$ into B such that $B^m B^n \subset B^{m+n}$ for $m, n \in Z$. Graded subalgebras and ideals are self-evident. A homomorphism of a graded algebra B into C is a homogeneous linear map l of degree zero of B into C such that $l(b_1 b_2) = l(b_1)l(b_2)$ for all $b_1, b_2 \in B$.

A graded algebra B is associative if $(b_1 b_2)b_3 = b_1(b_2 b_3)$ for all $b_1, b_2, b_3 \in B$. Such a B is commutative (anticommutative) if there holds for every pair of homogeneous elements $b_k \in B^{n_k}$, $k = 1, 2$, $b_1 b_2 = (-1)^{n_1 n_2} b_2 b_1$ $(b_1 b_2 = -(-1)^{n_1 n_2} b_2 b_1)$.

A *graded Lie algebra* is an graded anticommutative algebra which satisfies a graded version of the classical Jacobi identity. Precisely, a graded Lie algebra is a graded vector space $B = \oplus_{k \in Z} B^k$ together with a bilinear map $(b_1, b_2) \to [b_1, b_2]$ of $B \times B$ into B which satisfies the following conditions:

(I) $[B^m, B^n] \subset B^{m+n}$.

(II) If $b_1 \in B^{n_1}$, $b_2 \in B^{n_2}$, then $[b_1, b_2] = -(-1)^{n_1 n_2}[b_2, b_1]$.

(III) If $b_k \in B^{n_k}$, $k = 1, 2, 3$, then

$$(-1)^{n_1 n_3}[b_1, [b_2, b_3]] + (-1)^{n_2 n_1}[b_2, [b_3, b_1]] + (-1)^{n_3 n_2}[b_3, [b_1, b_2]] = 0.$$

The identity in (III) is called the Jacobi identity. When (II) is satisfied, it can be written in the following equivalent forms:

(III′) $[b_1, [b_2, b_3]] = [[b_1, b_2], b_3] + (-1)^{n_1 n_2}[b_2, [b_1, b_3]]$.

(III″) $[b_1, [b_2, b_3]] - (-1)^{n_1 n_2}[b_2, [b_1, b_3]] = [[b_1, b_2], b_3]$.

2.1.2 The Derivation Algebra

Much in the same way as the commutator defines a Lie algebra on an associative algebra, so the graded commutator defines a graded Lie algebra on an associative graded algebra. Precisely, if $B = \oplus_{k \in Z} B^k$ is an associative graded algebra, then the underlying graded vector space of B with the bracket determined by

$$[b_1, b_2] = b_1 b_2 - (-1)^{n_1 n_2} b_2 b_1 \tag{2.1}$$

for $b_k \in B^{n_k}$, $k = 1, 2$, is a graded Lie algebra.

As a particular example, let $E = \oplus_{k\in Z} E^k$ be a graded vector space. Then we have a natural structure of graded associative algebra on $End(E) = \oplus_{k\in Z} End^k(E)$, where $End^k(E)$ consists of linear endmorphisms of homogeneous degree k. We now further suppose that E is a graded algebra. Let $D : E \to E$ be an element of $End^k(E)$. We call D a k-derivation of the graded algebra E if there holds for any pair $e_1 \in E^{n_1}$ and $e_2 \in E^{n_2}$,

$$D(e_1 e_2) = (De_1)e_2 + (-1)^{kn_1} e_1(De_2). \tag{2.2}$$

The set $D^k(E)$ of all k-derivations is a subspace of $End^k(E)$. Let $D(E)$ denote the sum of the family $\{D^k(E)\}_{k\in Z}$; $D(E)$ is a graded subspace of $End(E)$. The commutator $[D_1, D_2]$ of two derivations of degree n_1 and n_2 is an (n_1+n_2)-derivation. Therefore, $D(E)$ is a graded subalgebra of the graded Lie algebra $End(E)$. We will call this graded Lie algebra $D(E)$ the *derivation algebra* of the graded algebra E.

A k-derivation D becomes a k-differential if it satisfies $D^2 = 0$, which is equivalent to $[D, D] = 0$ when k is an even number. A graded Lie algebra with specification of a 1-differential is an example of differential graded Lie algebras.

Let $B = \oplus_{k\in Z} B^k$ be a graded Lie algebra. Consider the adjoint representation. For any $b \in B^k$, the (III′) means $ad(b)$ is a k-derivation of B. (III″) further gives

$$[ad(b_1), ad(b_2)] = ad[b_1, b_2]$$

for $b_1 \in B^{n_1}$ and $b_2 \in B^{n_2}$. Hence adB is a graded Lie subalgebra of the derivation algebra $D(B)$. B becomes a differential graded Lie algebra with specification of an element $b \in B^1$ satisfying $[b, b] = 0$; the 1-differential is adb.

2.1.3 The Right Pre-Lie Algebra

The commutator also generates graded Lie algebras from *right pre-Lie algebras*. A graded algebra $B = \oplus_{k\in Z} B^k$ is called a right pre-Lie algebra if there holds for all $b_k \in B^{n_k}$, $k = 1, 2, 3$

$$b_1(b_2 b_3) - (b_1 b_2) b_3 = (-1)^{n_2 n_3}(b_1(b_3 b_2) - (b_1 b_3) b_2). \tag{2.3}$$

If B is a right pre-Lie algebra, then the underlying graded vector space B with bracket determined by

$$\{b_1, b_2\} = b_2 b_1 - (-1)^{n_1 n_2} b_1 b_2 \tag{2.4}$$

for $b_k \in B^{n_k}$, $k = 1, 2$ is a graded Lie algebra. The usual commutator

$$[b_1, b_2] = b_1 b_2 - (-1)^{n_1 n_2} b_2 b_1$$

also defines a graded Lie algebra on B.

2.1.4 The Semidirect Product

In this thesis, we will use the following notion of *semidirect product* for a graded Lie algebra.

Let B and C be graded Lie algebras. If there is a graded Lie algebra homomorphism

$$\Im : B \to D(C),$$

then we say that B acts on C through $\Im$. Given an action of B on C through $\Im$, the graded vector space $B \oplus C$ equipped with the bracket determined by

$$[(b_1, c_1), (b_2, c_2)] = ([b_1, b_2], \Im(b_1)c_2 - (-1)^{n_1 n_2} \Im(b_2)c_1 + [c_1, c_2]) \tag{2.5}$$

for $b_1 \in B^{n_1}$, $b_2 \in B^{n_2}$, $c_1 \in C^{n_1}$ and $c_2 \in C^{n_2}$, is a graded Lie algebra.

This graded Lie algebra structure $B \oplus C$ is the semidirect product of B and C and we denote it as $B \ltimes C$.

Note that B is a subalgebra of $B \ltimes C$ and that C is an ideal of $B \ltimes C$. This is actually the characteristic property of semidirect product. Precisely, if there is a graded Lie algebra structure on $B \oplus C$ such that B is a subalgebra and that C is an ideal, then there exists an action of B on C and as a graded Lie algebra, $B \oplus C$ is isomorphic to the semidirect product $B \ltimes C$ which is induced by this action.

2.1.5 The Knit Product

We now introduce a notion similar to the semidirect product, the *knit product* of two graded Lie algebras.

Let B and C be graded Lie algebras. A *derivatively knitted pair* of representations (α, β) for B and C are graded Lie algebra homomorphisms

$$\alpha : B \to End(C)$$
$$\beta : C \to End(B)$$

such that

$$\begin{aligned}
\alpha(b)[c_1, c_2] &= [\alpha(b)c_1, c_2] + (-1)^{nm_1}[c_1, \alpha(b)c_2] \\
&\quad -((-1)^{nm_1}\alpha(\beta(c_1)b)c_2 - (-1)^{(n+m_1)m_2}\alpha(\beta(c_2)b)c_1) \\
\beta(c)[b_1, b_2] &= [\beta(c)b_1, b_2] + (-1)^{mn_1}[b_1, \beta(c)b_2] \\
&\quad -((-1)^{mn_1}\beta(\alpha(b_1)c)b_2 - (-1)^{(m+n_1)n_2}\beta(\alpha(b_2)c)b_1)
\end{aligned} \tag{2.6}$$

for $b \in B^n$, $b_1 \in B^{n_1}$, $b_2 \in B^{n_2}$, $c \in C^m$, $c_1 \in C^{m_1}$ and $c_2 \in C^{m_2}$.

If there is a derivatively knitted pair (α, β) for B and C, then the graded vector space $B \oplus C$ equipped with the bracket determined by

$$[(b_1, c_1), (b_2, c_2)] = ([b_1, b_2] + \beta(c_1)b_2 - (-1)^{n_1 n_2}\beta(c_2)b_1, [c_1, c_2] + \alpha(b_1)c_2 - (-1)^{n_1 n_2}\alpha(b_2)c_1) \tag{2.7}$$

for $b_1 \in B^{n_1}$, $b_2 \in B^{n_2}$, $c_1 \in C^{n_1}$ and $c_2 \in C^{n_2}$, is a graded Lie algebra.

This graded Lie algebra $B \oplus C$ is called the knit product of B and C, and denoted as $B \bowtie C$.

Note that, when $\beta = 0$, the *knit product* degenerates to a semi-direct product.

The characteristic property of the knit product is that both B and C are subalgebras of $B \oplus C$. Precisely, if there is a graded Lie algebra structure on $B \oplus C$ such that both B and C are graded Lie subalgebras, then there is a derivatively knitted pair of representations for B and C such that, as a graded Lie algebra, $B \oplus C$ is isomorphic to the knit product of B and C induced by this pair.

2.2 The Shuffle Algebra

We introduce the shuffle algebra and notations about multi-shuffles in this section. These notations will be used in sequel and the basic properties of the shuffle algebra are helpful for us to deal with some complicated computations. The shuffle algebra was studied in [R] and in many subsequent works. Our exposition follows [AG].

Denote Σ_n for the symmetric group of $\{1, 2, \cdots, n\}$. A (p,q)-*shuffle* σ is an element of Σ_{p+q} which satisfies

$$\sigma(i) < \sigma(i+1), \quad i \neq p.$$

We will denote $sh(p,q)$ for the set of (p,q)-shuffles.

Consider the group algebras $R(\Sigma_n)$, $n = 0,1,2,\cdots$. We formulate a graded vector space

$$R(\Sigma) = \oplus_{n\geq 0} R(\Sigma_n).$$

On this graded vector space, defining

$$\Delta * \eta = \sum_{\sigma\in sh(m,n)} (-1)^{\sigma} \sigma \circ (\Delta \times \eta)$$

for $\Delta \in \Sigma_m$ and $\eta \in \Sigma_n$, with

$$(\Delta \times \eta)(i) = \begin{cases} \Delta(i) & 1 \leq i \leq m \\ \eta(i-m)+m & m+1 \leq i \leq m+n, \end{cases}$$

we get a graded algebra.

The subalgebra of $R(\Sigma)$ generated by $\mathbf{1}_k$, $k = 0,1,2,\cdots$, is called the *shuffle algebra*. The shuffle algebra is a graded commutative associative algebra, i.e, the following two identities hold,

$$\mathbf{1}_m * \mathbf{1}_n = (-1)^{mn} \mathbf{1}_n * \mathbf{1}_m \,, \tag{2.8}$$

$$(\mathbf{1}_m * \mathbf{1}_n) * \mathbf{1}_q = \mathbf{1}_m * (\mathbf{1}_n * \mathbf{1}_q) \,. \tag{2.9}$$

In this thesis, we will also use shuffles with more than two entries. For $n_1, \cdots, n_k$, positive integers, we denote

$$sh(n_1, \cdots, n_k)$$

for the set of $\sigma \in \Sigma_{n_1+\cdots+n_k}$ satisfying

$$\sigma(i) < \sigma(i+1), \quad i \neq n_1, n_1+n_2, \cdots, n_1+\cdots+n_{(k-1)} \,.$$

We will use notations

$$\begin{aligned}
sh_1(n_1,\cdots,n_k) &= \{\sigma \in sh(n_1,\cdots,n_k) | \sigma(1) = 1\}\,, \\
sh_2(n_1,\cdots,n_k) &= \{\sigma \in sh(n_1,\cdots,n_k) | \sigma(n_1+1) = 1\}\,, \\
&\vdots \\
sh_i(n_1,\cdots,n_k) &= \{\sigma \in sh(n_1,\cdots,n_k) | \sigma(n_1+n_2+\cdots+n_{i-1}+1) = 1\}\,, \\
&\vdots \\
sh_k(n_1,\cdots,n_k) &= \{\sigma \in sh(n_1,\cdots,n_k) | \sigma(n_1+\cdots+n_{k-1}+1) = 1\}\,.
\end{aligned}$$

For examples,

$$\begin{aligned}
sh(2,3) &= \{(12345),(13245),(14235),(15234),(23145),\\
&\quad (24135),(25134),(34125),(35124),(45123)\},\\
sh_1(2,3) &= \{(12345),(13245),(14235),(15234)\},\\
sh_2(2,3) &= \{(23145),(24135),(25134),(34125),(35124),(45123)\}.
\end{aligned}$$

Note that $sh_1(2,3)\bigcap sh_2(2,3) = \emptyset$ and $sh(2,3) = sh_1(2,3)\bigcup sh_2(2,3)$. Generally, we have

$$sh_i(n_1,\cdots,n_k)\bigcap sh_j(n_1,\cdots,n_k) = \emptyset \quad \text{for } i \neq j,$$

$$sh(n_1,\cdots,n_k) = \bigcup_{i=1}^{k} sh_i(n_1,\cdots,n_k).$$

For any object

$$X = (X_1, X_2, \cdots, X_{n_1+n_2+\cdots+n_k})$$

and $\sigma \in sh(n_1,\cdots,n_k)$, we use multi-index abbreviations in the following manner:

$$\begin{aligned}
X_{\sigma^1} &= (X_{\sigma(1)},\cdots,X_{\sigma(n_1)}),\\
X_{\sigma^2} &= (X_{\sigma(n_1+1)},\cdots,X_{\sigma(n_1+n_2)}),\\
&\ \vdots\\
X_{\sigma^i} &= (X_{\sigma(n_1+n_2+\cdots+n_{i-1}+1)},\cdots,X_{\sigma(n_1+n_2+\cdots+n_i)}),\\
&\ \vdots\\
X_{\sigma^k} &= (X_{\sigma(n_1+n_2+\cdots+n_{k-1}+1)},\cdots,X_{\sigma(n_1+n_2+\cdots+n_k)}).
\end{aligned}$$

For a fixed i, $\sigma \in sh_i(n_1,\cdots,n_k)$, the notations $X_{\sigma^1},\cdots,X_{\sigma^{i-1}},X_{\sigma^{i+1}},\cdots,X_{\sigma^k}$ are still as above. However, X_{σ^i} will mean $(X_{\sigma(n_1+n_2+\cdots+n_{i-1}+2)},\cdots,X_{\sigma(n_1+n_2+\cdots+n_i)})$.

2.3 Classical Examples of the Graded Lie Algebra

We introduce some classical examples of a graded Lie algebra. They will be used in later chapters.

2.3.1 The Schouten-Nijenhuis Algebra for a Lie Algebra

Let V be a finite-dimensional Lie algebra. The underlying graded vector space of the *Schouten-Nijenhuis algebra for the Lie algebra* V is

$$\bigwedge V[1] = \bigoplus_{k\geq 0} \bigwedge^{k+1} V.$$

Its graded Lie bracket is determined by

$$\begin{aligned}
&[X_1 \bigwedge \cdots \bigwedge X_{k_1}, Y_1 \bigwedge \cdots \bigwedge Y_{k_2}]_{\mathrm{SN}} \\
= &\sum_{i,j} (-1)^{i+j} [X_i, Y_j] \bigwedge X_1 \cdots \bigwedge \widehat{X_i} \bigwedge \cdots X_{k_1} \bigwedge Y_1 \bigwedge \cdots \bigwedge \widehat{Y_j} \bigwedge \cdots \bigwedge Y_{k_2},
\end{aligned} \tag{2.10}$$

where $X_s, s = 1, \cdots, k_1$ and $Y_t, t = 1, \cdots, k_2$ are all in V.

Two remarks are ready to make here for this Schouten-Nijenhuis algebra. First, since it also establishes that

$$[S_1, S_2 \bigwedge S_3]_{SN} = [S_1, S_2]_{SN} \bigwedge S_3 + (-1)^{k_1(k_2+1)} S_2 \bigwedge [S_1, S_3]_{SN}$$

for $S_i \in \bigwedge^{k_i+1} V$, $i = 1, 2, 3$, the Schouten-Nijenhuis algebra for a Lie algebra is an example of the Gerstenhaber algebra ([G1, 2, 3]), which attracts a great deal of attention recently in the mathematical physics community. Second, the degree 1, bracket-square 0 elements of this Schouten-Nijenhuis algebra are exactly the anti-symmetric r-matrices of the Lie algebra V. This was observed by Drinfeld ([Dr1]), Gelfand and Dorfman ([GD]).

In this thesis, we will also call this Schouten-Nijenhuis algebra the algebraic Schouten-Nijenhuis algebra.

2.3.2 The Schouten-Nijenhuis Algebra over a Manifold

Let N be a smooth manifold. The underlying graded vector space of the *Schouten-Nijenhuis algebra over the manifold* N is the graded vector space of polyvector fields on N,

$$\bigwedge TN = \bigoplus_{k\geq -1} \Gamma(\bigwedge^{k+1} TN),$$

where TN is the tangent bundle of N and $\Gamma(\bigwedge^{k+1} TN)$ is the space of sections of the exterior algebra bundle $\bigwedge^{k+1} TN$. The graded Lie bracket of this algebra is determined by (2.10) with $X_s, Y_t \in \mathbf{X}(N)$, together with $[f,g]_{SN} = 0$ for $f, g \in C^\infty(N)$ and $[X,f]_{SN} = Xf$.

This is the original algebra of Schouten ([S]) and Nijenhuis ([N1]) (See [MR] and [V1] for a modern exposition).

The Schouten-Nijenhuis algebra over a manifold N is also an example of the Gerstenhaber algebra. Its degree 1, bracket-square 0 elements are Poisson structures on N. Precisely, if $S \in \Gamma(\bigwedge^2 TN)$ satisfies $[S,S]_{SN} = 0$, then $\{f,g\} = S(df, dg)$ defines a Poisson bracket on N, and any Poisson structure on N is of this form.

Sometimes, we will call this Schouten-Nijenhuis algebra the geometric Schouten- Nijenhuis algebra.

2.3.3 The Nijenhuis-Richardson Algebra

Let V be a finite-dimensional vector space. Denote by $Alt^k(V,V)$ the space of k-linear alternating morphisms from $V \times \cdots \times V$ (k factors of V) to V and let $Alt^0(V,V) = V$. The *Nijenhuis-Richardson algebra* ([NR2]) is defined on the graded vector space

$$Alt(V,V)[1] = \bigoplus_{k \geq 0} Alt^{k+1}(V,V).$$

We need the concept of the composition product to introduce the graded Lie bracket of the NR algebra. For $P_i \in Alt^{k_i+1}(V,V)$, $i = 1,2$, their *composition product* $P_1P_2 \in Alt^{k_1+k_2+1}(V,V)$ is

$$P_1P_2(X_1, \cdots, X_{k_1+k_2+1}) = \sum_{\sigma \in sh(k_2+1, k_1)} (-1)^\sigma P_1(P_2(X_{\sigma^1}), X_{\sigma^2}). \tag{2.11}$$

The graded Lie bracket of the Nijenhuis-Richardson algebra is determined by

$$[P_1, P_2]_{\mathrm{NR}} = P_2P_1 - (-1)^{k_1k_2} P_1P_2. \tag{2.12}$$

Denote V^* for the dual space of V. Consider the graded vector space

$$\bigwedge V^* \bigotimes V = \bigoplus_{k \geq 0} (\bigwedge^k V^* \bigotimes V).$$

Since V is finite-dimensional, we have a graded vector space isomorphism

$$Alt(V,V) = \bigwedge V^* \bigotimes V. \tag{2.13}$$

This provides us an equivalent description for the Nijenhuis-Richardson algebra on $\bigwedge V^* \otimes V[1]$. In terms of $\bigwedge V^* \otimes V$, the composition product is

$$(\mu_1 \bigotimes X_1)(\mu_2 \bigotimes X_2) = \mu_2 \bigwedge i_{X_2}\mu_1 \bigotimes X_1 \tag{2.14}$$

where $\mu_i \otimes X_i \in \bigwedge^{k_i+1} V^* \otimes V, i = 1,2,$ and i_{X_2} is the usual insertion operator, and we can get from this the corresponding graded Lie bracket.

The fact that the bracket (2.12) defines a graded Lie algebra can be easily proved with this equivalent description. In fact, using (2.14), we can establish through a direct calculation the so called commutative-associative law,

$$P_1(P_2P_3) - (P_1P_2)P_3 = (-1)^{k_2k_3}(P_1(P_3P_2) - (P_1P_3)P_2), \tag{2.15}$$

where $P_i \in Alt^{k_i+1}(V,V), i = 1,2,3$. Therefore, $Alt(V,V)[1]$ with the composition product is a right pre-Lie algebra.

We make two remarks here. First, when V is infinite-dimensional, the Nijenhuis-Richardson algebra can still be defined on $Alt(V,V)[1]$. In this case, we can use (2.8) and (2.9) to prove the bracket determined by (2.11) and (2.12) is a graded Lie bracket. We will use this version of the Nijenhuis- Richardson algebra in Chapter 7. Second, some authors consider the Nijenhuis- Richardson algebra as an algebra structure on $Alt(V,V)$ rather than on $Alt(V,V)[1]$. Our choice is to make its properties to be stated in a more neat way.

2.3.4 The Lie Induced Algebra

Let V be a finite-dimensional vector space and W be a finite dimensional Lie algebra. Let $Alt^k(V,W)$ denotes the space of k-linear alternating morphisms from $\underbrace{V \bigotimes \cdots \bigotimes V}_{k}$ to W. On the graded vector space

$$Alt(V,W) = \bigoplus_{k \geq 0} Alt^k(V,W),$$

the bracket determined by

$$[E_1, E_2]_{\mathrm{LI}}(X_1, \cdots, X_{k_1+k_2}) = \sum_{\sigma \in sh(k_1,k_2)} (-1)^{\sigma}[E_1(X_{\sigma^1}), E_2(X_{\sigma^2})] \tag{2.16}$$

for $E_i \in Alt^{k_i}(V, W), i = 1, 2$, defines a graded Lie algebra.

This graded Lie algebra was used in [NR3] without a name. Since it is essentially induced by the Lie algebra structure on vector space W, we would like to call it the *Lie induced algebra* associated with V and W.

An equivalent description of the Lie induced algebra can be given through the following identification

$$Alt(V, W) = \bigwedge V^* \bigotimes W.$$

Here, we denote

$$\bigwedge V^* \bigotimes W = \bigoplus_{k \geq 0} \bigwedge^k V^* \bigotimes W.$$

In terms of $\bigwedge V^* \bigotimes W$, (2.16) is equivalent to

$$[v_1 \bigotimes Y_1, v_2 \bigotimes Y_2]_{\mathrm{LI}} = v_1 \bigwedge v_2 \bigotimes [Y_1, Y_2], \tag{2.17}$$

for $v_i \bigotimes Y_i \in \bigwedge^{k_i} V^* \bigotimes W, i = 1, 2$.

For any commutative COalgera C and Lie algebra W, there is a natural graded Lie algebra structure on $Hom(C, W)$. Consider $C = \bigwedge V$ with its exterior COalgebra structure, then the Lie induced algebra $Alt(V, W)$ is just a special case of this general construction since we clearly have $Alt(V, W) = Hom(\bigwedge V, W)$. For further information, we refer to [SS].

Following [N2], we will call the Lie induced algebra in the case $W = V$ the *cup algebra* for the Lie algebra V.

When both V and W are infinite-dimensional, the Lie induced algebra can still be defined on $Alt(V, W)$. In this case we can use (2.8) and (2.9) to show the bracket (2.16) defines a graded Lie bracket.

Chapter 3

One Structure and Some Operators

This is a preparatory chapter. We construct a semidirect product structure of the Nijenhuis-Richardson algebra and the Lie induced algebra. This structure will be used in Chapter 4, 5 and 7. The other part of this chapter is devoted to the study of the cup algebra. The focus is on its relations with two natural operators δ and Θ. These relations will play an important role in Chapter 4 and Chapter 5.

3.1 The Semidirect Product Structure

In this section we establish the semi-direct product structure of the Nijenhuis-Richardson algebra $Alt(V,V)[1]$ and the Lie induced algebra $Alt(V,W)$.

Let $P \in Alt^{l+1}(V,V), E \in Alt^k(V,W)$. We define

$$\Im(P)E(X_1,\cdots,X_{k+l}) = \sum_{\sigma\in sh(l+1,k-1)} (-1)^\sigma E(P(X_{\sigma^1}),X_{\sigma^2}). \tag{3.1}$$

It is clear that $\Im(P)E \in Alt^{k+l}(V,W)$. Hence, $\Im$ induces a morphism

$$\Im : Alt(V,V)[1] \longrightarrow End(Alt(V,W)).$$

The following proposition states $\Im$ actually defines a graded Lie algebra action of $Alt(V,V)[1]$ on $Alt(V,W)$.

Proposition 3.1

(3.1.a) $\Im(P) \in D^l(Alt(V,W))$.

(3.1.b) $\Im : Alt(V,V)[1] \longrightarrow D(Alt(V,W))$ *is a graded Lie algebra homomorphism.*

Proof. Let $E_i \in Alt^{k_i}(V, W)$, i=1, 2. In order to prove (3.1.a), we have to show

$$\Im(P)[E_1, E_2]_{\mathrm{LI}} = [\Im(P)E_1, E_2]_{\mathrm{LI}} + (-1)^{lk_1}[E_1, \Im(P)E_2]_{\mathrm{LI}}\,. \tag{3.2}$$

Without loss of generality, we assume

$$\begin{aligned} P &= \mu \bigotimes X\,, \\ E_i &= v_i \bigotimes Y_i, \quad i = 1, 2. \end{aligned}$$

Then

$$\begin{aligned} & \Im(P)[E_1, E_2]_{\mathrm{LI}} \\ = {} & \mu \bigwedge i_X(v_1 \bigwedge v_2) \bigotimes [Y_1, Y_2] \\ = {} & \mu \bigwedge i_X v_1 \bigwedge v_2 \bigotimes [Y_1, Y_2] + (-1)^{k_1} \mu \bigwedge v_1 \bigwedge i_X v_2 \bigotimes [Y_1, Y_2] \\ = {} & [\Im(P)E_1, E_2]_{\mathrm{LI}} + (-1)^{k_1+(l+1)k_1} v_1 \bigwedge \mu \bigwedge i_X v_2 \bigotimes [Y_1, Y_2] \\ = {} & [\Im(P)E_1, E_2]_{\mathrm{LI}} + (-1)^{lk_1}[E_1, \Im(P)E_2]_{\mathrm{LI}}\,. \end{aligned}$$

Note that in the above calculation we use

$$\Im(P)E_i = \mu \bigwedge i_X v_i \bigotimes Y_i, \quad i = 1, 2.$$

Now, let $P_i \in Alt^{k_i+1}(V, V), i = 1, 2$, and $E \in Alt^l(V, W)$. In order to prove (3.1.b), we have to show

$$[\Im(P_1), \Im(P_2)]E = \Im([P_1, P_2]_{\mathrm{NR}})E\,. \tag{3.3}$$

Without loss of generality, we can suppose

$$\begin{aligned} P_i &= \mu_i \bigotimes X_i, \quad i = 1, 2, \\ E &= v \bigotimes Y. \end{aligned}$$

Then

$$\begin{aligned} & [\Im(P_1), \Im(P_2)]E \\ = {} & \Im(P_1)\Im(P_2)E - (-1)^{k_1k_2}\Im(P_2)\Im(P_1)E \\ = {} & \Im(\mu_1 \bigotimes X_1)(\mu_2 \bigwedge i_{X_2} v \bigotimes Y) - (-1)^{k-1k_2}\Im(\mu_2 \bigotimes X_2)(\mu_1 \bigwedge i_{X_1} v \bigotimes Y) \\ = {} & \mu_1 \bigwedge i_{X_1}(\mu_2 \bigwedge i_{X_2} v) \bigotimes Y - (-1)^{k_1k_2} \mu_2 \bigwedge i_{X_2}(\mu_1 \bigwedge i_{X_1} v) \bigotimes Y \end{aligned}$$

$$
\begin{aligned}
&= \mu_1 \bigwedge i_{X_1}\mu_2 \bigwedge i_{X_2} v \bigotimes Y + (-1)^{k_2+1}\mu_1 \bigwedge \mu_2 \bigwedge i_{X_1} i_{X_2} v \bigotimes Y \\
&\quad -(-1)^{k_1 k_2}\mu_2 \bigwedge i_{X_2}\mu_1 \bigwedge i_{X_1} v \bigotimes Y + (-1)^{k_1(k_2+1)}\mu_2 \bigwedge \mu_1 \bigwedge i_{X_2} i_{X_1} v \bigotimes Y \\
&= \mu_1 \bigwedge i_{X_1}\mu_2 \bigwedge i_{X_2} v \bigotimes Y - (-1)^{k_1 k_2}\mu_2 \bigwedge i_{X_2}\mu_1 \bigwedge i_{X_1} v \bigotimes Y .
\end{aligned}
$$

On the other hand,

$$[P_1, P_2]_{\mathrm{NR}} = \mu_1 \bigwedge i_{X_1}\mu_2 \bigotimes X_2 - (-1)^{k_2} i_{X_2}\mu_1 \bigwedge \mu_2 \bigotimes X_1 .$$

Therefore,

$$
\begin{aligned}
&\Im([P_1, P_2]_{\mathrm{NR}})E \\
&= \mu_1 \bigwedge i_{X_1}\mu_2 \bigwedge i_{X_2} v \bigotimes Y - (-1)^{k_2} i_{X_2}\mu_1 \bigwedge \mu_2 \bigwedge i_{X_1} v \bigotimes Y \\
&= \mu_1 \bigwedge i_{X_1}\mu_2 \bigwedge i_{X_2} v \bigotimes Y - (-1)^{k_1 k_2}\mu_2 \bigwedge i_{X_2}\mu_1 \bigwedge i_{X_1} v \bigotimes Y .
\end{aligned}
$$

(3.3) is now clear. □

From §2.1.4, we have that the action $\Im$ generates a semi-direct product of the Nijenhuis-Richardson algebra $Alt(V,V)[1]$ and the Lie induced algebra $Alt(V,W)$. This is the graded Lie algebra in the following theorem.

Theorem 3.2 *On the graded vector space*

$$SP(V,W) = Alt(V,V)[1] \bigoplus Alt(V,W),$$

the bracket determined by

$$
\begin{aligned}
&[(P_1, E_1), (P_2, E_2)]_{SP} \\
&= ([P_1, P_2]_{NR}, [E_1, E_2]_{LI} + \Im(P_1)E_2 - (-1)^{k_1 k_2}\Im(P_2)E_1)
\end{aligned}
\tag{3.4}
$$

defines a graded Lie algebra, where $P_i \in Alt^{k_i+1}(V,V)$ and $E_i \in Alt^{k_i}(V,V), i = 1,2$.

For simplicity we will write

$$SP(V,W) = \bigoplus SP^k(V,W).$$

with the self-evident understanding

$$SP^k(V,W) = Alt^{k+1}(V,W) \bigoplus Alt^k(V,W).$$

The notation $SP(V) = SP(V,V)$ will also be used.

To end this section, we point out when V and W are both infinite-dimensional, Theorem 3.2 still holds. However, a different proof should be used. For such a proof, we can use (2.8) and (2.9).

3.2 Two Operators and the Cup Algebra

In this section, we introduce the operators δ and Θ, and discuss their interaction with the cup bracket.

3.2.1 Operators δ and Θ

Consider $\theta \in Alt^2(V,V)$. By definition, we have

$$\theta\theta(X_1,X_2,X_3) = \theta(\theta(X_1,X_2),X_3) + \theta(\theta(X_2,X_3),X_1) + \theta(\theta(X_3,X_1),X_2)\,.$$

Hence the bracket

$$[X_1,X_2] = \theta(X_1,X_2)$$

defines a Lie algebra if and only if $\theta^2 = 0$, which is also equivalent to $\{\theta,\theta\} = 0$.

Fix such a θ on V.

We recall that the Chavelley-Eilenberg([CE]) coboundary operator

$$\delta : Alt^k(V,V) \longrightarrow Alt^{k+1}(V,V)$$

for the adjoint representation of the Lie algebra V is defined by

$$\begin{aligned} & \delta Q(X_1,\cdots,X_{k+1}) \\ = \; & \sum_{i=1}^{k+1}(-1)^{i-1}[X_i, Q(X_1,\cdots,\widehat{X_i},\cdots,X_{k+1})] \\ & +\sum_{i<j}(-1)^{i+j}Q([X_i,X_j],X_1,\cdots,\widehat{X_i},\cdots,\widehat{X_j},\cdots,X_{k+1})\,. \end{aligned} \tag{3.5}$$

We also recall that coboundary operator for the trivial representation on V is defined by

$$D : Alt^k(V,V) \longrightarrow Alt^{k+1}(V,V)\,,$$

$$
\begin{aligned}
& DQ(X_1, \cdots, X_{k+1}) \\
= & \sum_{i<j}(-1)^{i+j} Q([X_i, X_j], X_1, \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, X_{k+1}).
\end{aligned}
\tag{3.6}
$$

For δ and D we have the following straight-forward result:

Proposition 3.3

(3.3.a) $\delta Q = -[\theta, Q]_{NR}$.

(3.3.b) $DQ = -Q\theta$.

Now we introduce one more operator, which will play a critical role in our study of the Kodaira-Spencer algebra. For $Q \in Alt^k(V, V)$, let us define

$$
\Theta Q = -\theta Q\,. \tag{3.7}
$$

It is clear that

$$
[\theta, Q]_{\mathrm{NR}} = Q\theta - (-1)^{k-1}\theta Q\,.
$$

Therefore, we have

$$
\Theta Q = (-1)^k(\delta - D)Q\,. \tag{3.8}
$$

We collect some identities associated with the above-introduced operators in the following,

Proposition 3.4

(3.4.a) $\delta^2 = 0$.

(3.4.b) $D^2 = 0$.

(3.4.c) $\Theta D = \delta\Theta$.

(3.4.d) $\Theta^2 = D\delta + \delta D$.

Proof. (3.4.a) follows directly from the graded Jacobi identity of the Nijenhuis-Richardson algebra.

By the commutative-associative law, we have

$$
Q(\theta^2) - (Q\theta)\theta = -(Q(\theta^2) - (Q\theta)\theta)\,.
$$

Since $\theta^2 = 0$, this gives us

$$-(Q\theta)\theta = (Q\theta)\theta\,.$$

Therefore,

$$(Q\theta)\theta = 0\,,$$

i.e., $D^2Q = 0$. This is (3.4.b).

For (3.4.c), again by the commutative-associative law, we have

$$\theta(Q\theta) - (\theta Q)\theta) = (-1)^{k-1}(\theta(\theta Q) - (\theta^2)Q)\,.$$

Since $\theta^2 = 0$, it gives us

$$\theta(Q\theta) = [\theta, \theta Q]_{\mathrm{NR}}\,.$$

Hence, $\Theta D = \delta\Theta$.

The last identity (3.4.d) follows from (3.8), (3.4.a) and (3.4.b). □

Proposition 3.4.a and 3.4.b are well-known. However, I have not seen their proofs based on the commutative-associative law before. As far as I know, the operator Θ is considered here for the first time.

3.2.2 The Cup Algebra

Recall the cup algebra is the Lie induced algebra with $W = V$. Here we express the cup bracket in terms of Θ and δ and prove some formulae regarding its behaviour under the action of D, δ and Θ. The formula associated with Θ is new.

Lemma 3.5 *Let $[\,]_C$ denote the bracket for the cup algebra, and $Q_i \in Alt^{k_i}(V, V), i = 1, 2$, then there holds $[Q_1, Q_2]_C = (-1)^{k_2}((\Theta Q_2)Q_1 - \Theta(Q_2Q_1))$.*

Proof. We apply the isomorphism

$$Alt(V, V) = \bigwedge V^* \bigotimes V\,.$$

Without loss of generality, we assume

$$\theta = \omega \bigotimes Y \in \bigwedge^2 V^* \bigotimes V\,.$$

Let $Q_i = v_i \bigotimes X_i, i = 1, 2$. We have

$$\begin{aligned}
& (\Theta Q_2)Q_1 \\
= \; & -(\theta Q_2)Q_1 \\
= \; & -(v_2 \bigwedge i_{X_2}\omega \bigotimes Y)(v_1 \bigotimes X_1) \\
= \; & -v_1 \bigwedge i_{X_1} v_2 \bigwedge i_{X_2}\omega \bigotimes Y + (-1)^{k_2}\omega(X_1, X_2) v_1 \bigwedge v_2 \bigotimes Y \, ;
\end{aligned}$$

and

$$\begin{aligned}
& \Theta(Q_2 Q_1) \\
= \; & -\theta(Q_2 Q_1) \\
= \; & -(\omega \bigotimes Y)(v_1 \bigwedge i_{X_1} v_2 \bigotimes X_2) \\
= \; & -v_1 \bigwedge i_{X_1} v_2 \bigwedge i_{X_2}\omega \bigotimes Y \, .
\end{aligned}$$

Therefore,

$$(\Theta Q_2)Q_1 - \Theta(Q_2 Q_1) = (-1)^{k_2}\omega(X_1, X_2) v_1 \bigwedge v_2 \bigotimes Y \, .$$

However, it is clearly true that

$$[Q_1, Q_2]_{\mathrm{C}} = \omega(X_1, X_2) v_1 \bigwedge v_2 \bigotimes Y \, . \square$$

Proposition 3.6 ([N2]) *Let* $Q_i, i = 1, 2$ *be as in the above lemma, we have* $[Q_1, Q_2]_C = (\delta Q_2)Q_1 - (-1)^{k_1} Q_2 \delta Q_1 + (-1)^{k_1}\delta(Q_2 Q_1)$.

Proof. We can prove it as follows:

$$\begin{aligned}
& (\delta Q_2)Q_1 - (-1)^k Q_2 \delta Q_1 + (-1)^{k_1}\delta(Q_2 Q_1) \\
= \; & -[\theta, Q_2]_{\mathrm{NR}} Q_1 + (-1)^{k_1} Q_2 [\theta, Q_1]_{\mathrm{NR}} - (-1)^{k_1}[\theta, Q_2 Q_1]_{\mathrm{NR}} \\
= \; & -(Q_2\theta)Q_1 + (-1)^{k_2 - 1}(\theta Q_2)Q_1 + (-1)^{k_1} Q_2(Q_1\theta) - (-1)^{k_1 + k_1 - 1} Q_2(\theta Q_1) \\
& -(-1)^{k_1}(Q_2 Q_1)\theta + (-1)^{k_1 + k_1 + k_2}\theta(Q_2 Q_1) \\
= \; & (-1)^{k_2}((-\theta Q_2)Q_1 - (-\theta)(Q_2 Q_1)) \\
= \; & (-1)^{k_2}((\Theta Q_2)Q_1 - \Theta(Q_2 Q_1)) \\
= \; & [Q_1, Q_2]_{\mathrm{C}} \, .
\end{aligned}$$

The last step use lemma 3.5. $\square$

Lemma 3.7 *Let $Q_i \in Alt^{k_i}(V,V), i = 1,2$. We have*

(3.7.a) $D[Q_1,Q_2]_C = [DQ_1,Q_2]_C + (-1)^{k_1}[Q_1,DQ_2]_C$.

(3.7.b) $\delta[Q_1,Q_2]_C = [\delta Q_1,Q_2]_C + (-1)^{k_1}[Q_1,\delta Q_2]_C$.

Proof. Without loss of generality, we assume $Q_i = v_i \bigotimes X_i$, then

$$\begin{aligned} & D[Q_1,Q_2]_C \\ = \;& D(v_1 \bigwedge v_2 \bigotimes [X_1,X_2]) \\ = \;& d(v_1 \bigwedge v_2) \bigotimes [X_1,X_2] \\ = \;& dv_1 \bigwedge v_2 \bigotimes [X_1,X_2] + (-1)^{k_1} v_1 \bigwedge dv_2 \bigotimes [X_1,X_2] \\ = \;& [DQ_1,Q_2]_C + (-1)^{k_1}[Q_1,DQ_2]_C\,. \end{aligned}$$

This is (3.7.a).

For (3.7.b), we use Proposition 3.6 and $\delta^2 = 0$.

$$\delta[Q_1,Q_2]_C = \delta((\delta Q_2)Q_1) - (-1)^{k_1}\delta(Q_2\delta Q_1)\,,$$

$$[\delta Q_1,Q_2]_C = \delta Q_2 \delta Q_1 + (-1)^{k_1+1}\delta(Q_2\delta Q_1)\,,$$

$$[Q_1,\delta Q_2]_C = (-1)^{k_1}\delta((\delta Q_2)Q_1) - (-1)^{k_1}(\delta Q_2)(\delta Q_1)\,.$$

Therefore,

$$\delta[Q_1,Q_2]_C = [\delta Q_1,Q_2]_C + (-1)^{k_1}[\delta Q_1,\delta Q_2]_C\,.\square$$

Lemma (3.7.a) and (3.7.b) shows that both operators D and δ endow the cup algebra $Alt(V,V)$ with differential graded Lie algebra structures. In later chapters, we will also use the fact that the Nijenhuis-Richardson algebra is also a differential graded Lie algebra, i.e.

$$\delta[P_1,P_2]_{NR} = [\delta P_1,P_2]_{NR} + (-1)^{k_1}[P_1,\delta P_2]_{NR}\,. \tag{3.9}$$

This identity is a direct consequence of the graded Jacobi identity of the algebra because of Proposition 3.6.a.

Proposition 3.8 *There holds*

$$\Theta[Q_1,Q_2]_C = (-1)^{k_2}[\Theta Q_1,Q_2]_C + [Q_1,\Theta Q_2]_C.$$

Proof. It is a consequence of Lemma 3.7 and the identity (3.8).

$$
\begin{aligned}
& \Theta[Q_1, Q_2]_{\mathrm{C}} \\
= \; & (-1)^{k_1+k_2}(\delta - D)[Q_1, Q_2]_{\mathrm{C}} \\
= \; & (-1)^{k_1+k_2}[(\delta - D)Q_1, Q_2]_{\mathrm{C}} + (-1)^{k_1+k_2+k_1}[Q_1, (\delta - D)Q_2]_{\mathrm{C}} \\
= \; & (-1)^{k_2}[(-1)^{k_1}(\delta - D)Q_1, Q_2]_{\mathrm{C}} + [Q_1, (-1)^{k_2}(\delta - D)Q_2]_{\mathrm{C}} \\
= \; & (-1)^{k_2}[\Theta Q_1, Q_2]_{\mathrm{C}} + [Q_1, \Theta Q_2]_{\mathrm{C}} . \square
\end{aligned}
$$

Chapter 4

The Frölicher-Nijenhuis Algebra

The Frölicher-Nijenhuis algebra was first defined in the geometric context by Frölicher and Nijenhuis in 1957([FN1]). Later the version we are concerned was studied by Professor Nijenhuis through a purely algebraic approach([N2]). In this chapter, we introduce Nijenhuis' idea with an emphasis on the knit product structure of the Nijenhuis-Richardson and the Frölicher-Nijenhuis algebras, which was not explicitly demonstrated in the original paper. New results include a different proof of an interesting formula recently discovered by Nijenhuis ([N3]).

4.1 Nijenhuis' Idea

We follow [N2] to introduce the Frölicher-Nijenhuis algebra associated with a Lie algebra.

Let V be a Lie algebra. Consider the graded vector space embedding

$$\begin{aligned} \tau : Alt(V,V) &\longrightarrow SP(V)\,, \\ \tau(Q) &= ((-1)^k \delta Q, Q)\,, \end{aligned} \tag{4.1}$$

where $Q \in Alt^k(V,V)$.

Lemma 4.1 *The image of τ is a subalgebra of $SP(V)$.*

Proof. For $Q_i \in Alt^{k_i}(V,V), i = 1,2$, we have

$$\begin{aligned} & [((-1)^{k_1}\delta Q_1, Q_1), ((-1)^{k_2}\delta Q_2, Q_2)]_{\mathrm{SP}} \\ = \;& ((-1)^{k_1+k_2}[\delta Q_1, \delta Q_2]_{\mathrm{NR}}, [Q_1, Q_2]_{\mathrm{C}} + (-1)^{k_1} Q_2 \delta Q_1 - (-1)^{k_2(k_1+1)} Q_1 \delta Q_2\,. \end{aligned}$$

It only needs to show

$$\delta((-1)^{k_1}Q_2\delta Q_1 - (-1)^{k_2(k_1+1)}Q_1\delta Q_2 + [Q_1, Q_2]_{\mathrm{C}}) = [\delta Q_1, \delta Q_2]_{\mathrm{NR}}. \tag{4.2}$$

However,

$$\begin{aligned}
& \delta([Q_1, Q_2]_{\mathrm{C}}) + (-1)^{k_1}\delta(Q_1\delta Q_1) - (-1)^{k_2(k_1+1)}\delta(Q_1\delta Q_2) \\
= \; & \delta((\delta Q_2)Q_1 - (-1)^{k_1}Q_2 \cdot \delta Q_1 + (-1)^{k_1}\delta(Q_2Q_1)) \\
& +(-1)^{k_1}\delta(Q_2\delta Q_1) - (-1)^{k_2(k_1+1)}\delta(Q_1\delta Q_2) \\
= \; & \delta((\delta Q_2)Q_1 - (-1)^{k_2(k_1+1)}Q_1(\delta Q_2)) \\
= \; & \delta[Q_1, \delta Q_2]_{\mathrm{NR}} \\
= \; & [\delta Q_1, \delta Q_2]_{\mathrm{NR}}.
\end{aligned}$$

Here we use Proposition 3.6 in the first step and (3.9) in the last step. This proves (4.2) and hence the lemma. □

This lemma can be reformulated as

Theorem 4.2 ([N2]) *The graded vector space $Alt(V, V)$ with bracket determined by*

$$[Q_1, Q_2]_{FN} = [Q_1, Q_2]_C + (-1)^{k_1}Q_2\delta Q_1 - (-1)^{k_2(k-1+1)}Q_1\delta Q_2 \tag{4.3}$$

is a graded Lie algebra and τ of (4.1) is an injective graded Lie algebra homomorphism.

The graded Lie algebra in this theorem is commonly called the *Frölicher-Nijenhuis algebra* for the Lie algebra V.

Applying Proposition 3.6, we have another expression for its graded Lie bracket,

$$[Q_1, Q_2]_{\mathrm{FN}} = (-1)^{k_1}\delta(Q_2Q_1) + [Q_1, \delta Q_2]_{\mathrm{NR}}. \tag{4.4}$$

In terms of the bracket (4.3), we can rewrite (4.2) as

$$\delta[Q_1, Q_2]_{\mathrm{FN}} = [\delta Q_1, \delta Q_2]_{\mathrm{NR}}. \tag{4.5}$$

This shows δ is a graded Lie algebra homomorphism. Note that the fact that δ is of degree 1 plays an important role here. Since the Frölicher-Nijenhuis bracket is of degree 0 and the Nijenhuis-Richardson bracket is essentially of degree 1, this makes δ a graded Lie algebra of degree 0.

From now on we will denote $Alt_{\mathrm{C}}(V,V)$ and $Alt_{\mathrm{FN}}(V,V)$ for the cup algebra and Frölicher-Nijenhuis algebra on the graded vector space $Alt(V,V)$, respectively.

The following proposition shows how the Frölicher-Nijenhuis bracket and the Nijenhuis-Richardson interact.

Proposition 4.3 *The morphisms*

$$\alpha : Alt(V,V)[1] \longrightarrow End(Alt(V,V)),$$
$$\beta : Alt_{FN}(V,V) \longrightarrow End(Alt(V,V)[1])$$

determined by

$$\begin{aligned} \alpha(P)Q &= QP, \\ \beta(Q)P &= [Q,P]_{FN} \end{aligned} \tag{4.6}$$

for $P \in Alt(V,V)[1]$ and $Q \in Alt(V,V)$ constitute a derivatively knitted pair of representations of the Nijenhuis-Richardson algebra and the Frölicher-Nijenhuis algebra, i.e.

(4.3.a) *α , β are graded Lie algebra homomorphisms.*

(4.3.b) *for $Q \in Alt^l(V,V)$, $Q_i \in Alt^{l_i}(V,V)$, $P \in Alt^{k+1}(V,V)$ and $P_i \in Alt^{k_i+1}(V,V)$, $i = 1,2$, we have*

$$\begin{aligned} [Q_1,Q_2]_{FN}P &= [Q_1P,Q_2]_{FN} + (-1)^{kl_1}[Q_1,Q_2P]_{FN} \\ &\quad -((-1)^{kl_1}Q_2[Q_1,P]_{FN} - (-1)^{(k+l_1)l_2}Q_1[Q_2,P]_{FN}), \end{aligned} \tag{4.7.1}$$

$$\begin{aligned} [Q,[P_1,P_2]_{NR}]_{FN} &= [[Q,P_1]_{FN},P_2]_{NR} + (-1)^{lk_1}[P_1,[Q,P_2]_{FN}]_{NR} - \\ &\quad ((-1)^{lk_1}[QP_1,P_2]_{FN} - (-1)^{(l+k_1)k_2}[P_2,QP_2]_{FN}). \end{aligned} \tag{4.7.2}$$

Proof. α is a graded Lie algebra homomorphism since the commutative-associative law holds; β is also a homomorphism since the graded Jacobi identity of the Frölicher-Nijenhuis algebra holds. A long but straight-forward calculation establishes (4.7.1) and (4.7.2). $\square$

A direct consequence of this proposition is

Theorem 4.4 *On the graded vector space $Alt(V,V)[1] \bigoplus Alt(V,V)$, the bracket*

$$\begin{aligned} & [(P_1,Q_1),(P_2,Q_2)]_{KP} \\ = \ & ([P_1,P_2]_{NR} + [Q_1,P_2]_{FN} - (-1)^{k_1k_2}[Q_2,P_1]_{FN}, \\ & [Q_1,Q_2]_{FN} + Q_2P_1 - (-1)^{k-1k_2}Q_1P_2), \end{aligned} \tag{4.8}$$

defines a graded Lie algebra structure.

It is the knit product of the Nijenhuis-Richardson algebra and the Frölicher-Nijenhuis algebra.

A more straight way to show that the bracket (4.8) defines a graded lie algebra structure on $Alt(V,V)[1] \bigoplus Alt(V,V)$ is as follows:

By (4.4) and (4.5), we have

$$\begin{aligned} & [(P_1 + (-1)^{k_1}\delta Q_1, Q_1),(P_2 + (-1)^{k_2}\delta Q_2, Q_2)]_{\mathrm{SP}} \\ = \ & ([P_1 + (-1)^{k_1}\delta Q_1, P_2 + (-1)^{k_2}\delta Q_2]_{\mathrm{NR}}, \\ & [Q_1,Q_2]_{\mathrm{C}} + Q_2(P_1 + (-1)^{k_1}\delta Q_1) - (-1)^{k_1k_2}Q_1(P_2 + (-1)^{k_2}\delta Q_2)) \\ = \ & ([P_1,P_2]_{\mathrm{NR}} + (-1)^{k_2}[P_1,\delta Q_2]_{\mathrm{NR}} + (-1)^{k_1}[\delta Q_1, P_2]_{\mathrm{NR}} + (-1)^{k_1+k_2}[\delta Q_1,\delta Q_2]_{\mathrm{NR}}, \\ & [Q_1,Q_2]_{\mathrm{C}} + (-1)^{k_1}Q_2\delta Q_1 - (-1)^{k_2(k_1+1)}Q_1\delta Q_2 + Q_2P_1 - (-1)^{k_1k_2}Q_1P_2) \\ = \ & ([P_1,P_2]_{\mathrm{NR}} + [Q_1,P_2]_{\mathrm{FN}} - (-1)^{k_1k_2}[Q_2,P_1]_{\mathrm{FN}} + \\ & (-1)^{k_1+k_2}\delta([Q_1,Q_2]_{\mathrm{FN}} + Q_2P_1 - (-1)^{k_1k_2}Q_1P_2), \\ & [Q_1,Q_2]_{\mathrm{FN}} + Q_2P_1 - (-1)^{k_1k_2}Q_1P_2), \end{aligned} \tag{4.9}$$

where $P_i \in Alt^{k_i+1}(V,V)$ and $Q_i \in Alt^{k_i}(V,V)$, $i = 1,2$. If we define

$$\begin{aligned} \hat{\tau} : Alt(V,V)[1] \bigoplus Alt(V,V) &\longrightarrow SP(V), \\ \hat{\tau}(P,Q) &= (P + (-1)^k\delta Q, Q), \end{aligned} \tag{4.10}$$

for $(P,Q) \in Alt^{k+1}(V,V) \bigoplus Alt^k(V,V)$ then it follows from (4.9)

$$
\begin{aligned}
&\hat{\tau}[(P_1, Q_1), (P_2, Q_2)]_{\mathrm{KP}} \\
= \; &[\hat{\tau}(P_1, Q_1), \hat{\tau}(P_2, Q_2)]_{\mathrm{SP}} .
\end{aligned}
\tag{4.11}
$$

Since $\hat{\tau}$ is a graded vector space isomorphis, the graded Jacobi identity for the bracket (4.8) follows from that of the bracket $[\ ]_{SP}$.

We note that $\hat{\tau}$ is actually a graded Lie algebra homomorphism.

In the end of this section, for comparison with results in later chapters, we give two other expressions of the bracket (4.3) here (cf. [KMS]).

The first one is

$$
\begin{aligned}
&[Q_1, Q_2]_{FN}(X_1, \cdots, X_{k_1+k_2}) \\
= &\sum_{\sigma \in sh(k_1, k_2)} (-1)^{\sigma} [Q_1(X_{\sigma^1}), Q_2(X_{\sigma^2})] \\
&- \sum_{\sigma \in sh(k_1, 1, k_2-1)} (-1)^{\sigma} Q_2([Q_1(X_{\sigma^1}), X_{\sigma(k_1+1)}], X_{\sigma^3}) \\
&+ (-1)^{k_1 k_2} \sum_{\sigma \in sh(k_2, 1, k_1-1)} (-1)^{\sigma} Q_1([Q_2(X_{\sigma^1}, X_{\sigma(k_2+1)}], X_{\sigma^3}) \\
&- (-1)^{k_1} \sum_{\sigma \in sh(2, k_1-1, k_2-1)} (-1)^{\sigma} Q_2(Q_1([X_{\sigma(1)}, X_{\sigma(2)}], X_{\sigma^2}), X_{\sigma^3}) \\
&+ (-1)^{k_2(k_1+1)} \sum_{\sigma \in sh(2, k_2-1, k_1-1)} (-1)^{\sigma} Q_1(Q_2([X_{\sigma(1)}, X_{\sigma(2)}], X_{\sigma^2}), X_{\sigma^3})
\end{aligned}
\tag{4.12}
$$

This follows directly from (4.3) and the definition of δ.

Consider the isomorphism

$$
Alt(V, V) = \bigwedge V^* \bigotimes V .
$$

Let $Q_i = v_i \bigotimes X_i, i = 1, 2$. Then the bracket (4.3) can also be expressed as

$$
\begin{aligned}
&[v_1 \bigotimes X_1, v_2 \bigotimes X_2]_{\mathrm{FN}} \\
= \; &v_1 \bigwedge v_2 \bigotimes [X_1, X_2] + v_1 \bigwedge ad^*_{X_1} v_2 \bigotimes X_2 - ad^*_{X_2} v_1 \bigwedge v_2 \bigotimes X_1
\end{aligned}
$$

$$
\begin{aligned}
&+(-1)^{k_1}(dv_1 \bigwedge i_{X_1}v_2 \bigotimes X_2 + i_{X_2}v_1 \bigwedge dv_2 \bigotimes X_1) \\
= \;& v_1 \bigwedge v_2 \bigotimes [X_1, X_2] - (i_{X_2}dv_1 \bigwedge v_2 \bigotimes X_1 - (-1)^{k_1k_2} i_{X_1}dv_2 \bigwedge v_1 \bigotimes X_2) \\
&-(d(i_{X_2}v_1 \bigwedge v_2) \bigotimes X_1 - (-1)^{k_1k_2} d(i_{X_1}v_2 \bigwedge v_1) \bigotimes X_2).
\end{aligned}
\tag{4.13}
$$

Here ad^* is the representation of V on $\bigwedge^{k_1} V^* (or \bigwedge^{k_2} V^*)$ induced by the adjoint representation, d is the Chevalley-Eilenberg coboundary operator associated with the trivial representation of V on R and we use

$$ad^*_X = i_X d + di_X,$$

which can be easily proved.

4.2 A Derived Bracket

We consider in this section a bracket derived from the Nijenhuis-Richardson bracket. This derived bracket includes the Frölicher-Nijenhuis bracket as a special case. We prove an interesting formula which reveals the relation between the Frölicher-Nijenhuis brackets of a Lie algebra and of its Nijenhuis deformations.

Let V be a vector space. Inspired by (4.4), we consider for a fixed $P \in Alt^2(V,V)$ the bracket determined by

$$[Q_1, Q_2]_{\mathrm{FN}(P)} = (-1)^{k_1}[P, Q_2Q_1]_{\mathrm{NR}} + [Q_1, [P, Q_2]_{\mathrm{NR}}]_{\mathrm{NR}} \tag{4.14}$$

for $Q_i \in Alt^{k_i}(V,V), i = 1,2$.

Note that when V is equipped with a Lie algebra as before, its Frölicher-Nijenhuis bracket is nothing but $[\ ]_{FN(-\theta)}$, i.e., $[\ ]_{FN} = [\ ]_{FN(-\theta)}$. Also, the argument in last section implies that the bracket (4.14) is a graded Lie algebra bracket on $Alt(V,V)$ if P satisfies $[P,P]_{NR} = 0$.

For $P \in Alt^2(V,V)$, $Q \in Alt^1(V,V)$, it is clear $[P,Q]_{\mathrm{NR}} \in Alt^2(V,V)$. Hence, we can consider the bracket $[\ ]_{FN([P,Q]_{\mathrm{NR}})}$ defined by (4.14) for $[P,Q]_{\mathrm{NR}}$. The central result in this section is the following theorem:

Theorem 4.5

$$\begin{aligned} & [Q_1, Q_2]_{FN([P,Q]_{NR})} \\ = \ & [[Q,Q_1]_{NR}, Q_2]_{FN(P)} + [Q_1, [Q,Q_2]_{NR}]_{FN(P)} - [Q, [Q_1,Q_2]_{FN(P)}]_{NR}. \end{aligned}$$

This theorem expresses the bracket $[\ ]_{FN([P,Q]_{\mathrm{NR}})}$ in terms of $[\ ]_{FN(P)}$and $[Q,\ ]_{NR}$.

We need a lemma to prove this theorem.

Lemma 4.6 *For $Q \in Alt^1(V,V), Q_i \in Alt^{k_i}(V,V), i = 1,2,$ we have*

$$[Q, Q_1Q_2]_{NR} = [Q,Q_1]_{NR}Q_2 + Q_1[Q,Q_2]_{NR}. \tag{4.15}$$

Proof. The critical point is that for $Q \in Alt^1(V,V)$ there holds

$$Q(Q_1Q_2) = (QQ_1)Q_2.$$

Therefore,

$$\begin{aligned}
& [Q, Q_1Q_2]_{\mathrm{NR}} - [Q, Q_1]_{\mathrm{NR}}Q_2 - Q_1[Q, Q_2]_{\mathrm{NR}} \\
= \ & (Q_1Q_2)Q - Q(Q_1Q_2) - (Q_1Q)Q_2 + (QQ_1)Q_2 - Q_1(Q_2Q) + Q_1(QQ_2) \\
= \ & (Q_1Q_2)Q - Q_1(Q_2Q) - (Q_1Q)Q_2 + Q_1(QQ_2) \\
= \ & 0\,.
\end{aligned}$$

The last step uses the commutative-associative law. □

Proof of Theorem 4.5. Applying Lemma 4.6 and the graded Jacobi identity of the Nijenhuis-Richardson algebra, we can make the following calculation:

$$\begin{aligned}
& [Q_1, Q_2]_{\mathrm{FN}([P,Q]_{\mathrm{NR}})} \\
= \ & (-1)^{k_1}[[P,Q]_{\mathrm{NR}}, Q_2Q_1]_{\mathrm{NR}} + [Q_1, [[P,Q]_{\mathrm{NR}}, Q_2]_{\mathrm{NR}}]_{\mathrm{NR}} \\
= \ & (-1)^{k_1}[P, [Q, Q_2Q_1]_{\mathrm{NR}}]_{\mathrm{NR}} - (-1)^{k_1}[Q, [P, Q_2Q_1]_{\mathrm{NR}}]_{\mathrm{NR}} \\
& +[Q_1, [P, [Q, Q_2]_{\mathrm{NR}}]_{\mathrm{NR}} - [Q, [P, Q_2]_{\mathrm{NR}}]_{\mathrm{NR}}]_{\mathrm{NR}} \\
= \ & (-1)^{k_1}[P, [Q, Q_2Q_1]_{\mathrm{NR}}]_{\mathrm{NR}} - (-1)^{k_1}[Q, [P, Q_2Q_1]_{\mathrm{NR}}]_{\mathrm{NR}} \\
& +[Q_1, [P, [Q, Q_2]_{\mathrm{NR}}]_{\mathrm{NR}}]_{\mathrm{NR}} \\
& +[[Q, Q_1]_{\mathrm{NR}}, [P, Q_2]_{\mathrm{NR}}]_{\mathrm{NR}} - [Q, [Q_1, [P, Q_2]_{\mathrm{NR}}]_{\mathrm{NR}}]_{\mathrm{NR}} \\
= \ & -[Q, [Q_1, Q_2]_{\mathrm{FN}(P)}]_{\mathrm{NR}} \\
& +[Q_1, [P, [Q, Q_2]_{\mathrm{NR}}]_{\mathrm{NR}}]_{\mathrm{NR}} + (-1)^{k_1}[P, [Q, Q_2]_{\mathrm{NR}}Q_1]_{\mathrm{NR}} \\
& +[[Q, Q_1]_{\mathrm{NR}}, [P, Q_2]_{\mathrm{NR}}]_{\mathrm{NR}} + (-1)^{k_1}[P, Q_2[Q, Q_1]_{\mathrm{NR}}]_{\mathrm{NR}} \\
& +(-1)^{k_1}[P, [Q, Q_2Q_1]_{\mathrm{NR}} - [Q, Q_2]_{\mathrm{NR}}Q_1 - Q_2[Q, Q_1]_{\mathrm{NR}}]_{\mathrm{NR}} \\
= \ & [[Q, Q_1]_{\mathrm{NR}}, Q_2]_{\mathrm{FN}(P)} + [Q_1, [Q, Q_2]_{\mathrm{NR}}]_{\mathrm{FN}(P)} - \\
& [Q, [Q_1, Q_2]_{\mathrm{FN}(P)}]_{\mathrm{NR}}\,. \square
\end{aligned}$$

In order to provide an application of Theorem 4.5, we now suppose V is equipped with a Lie algebra structure as before.

Let $Q \in Alt^1(V,V) = Hom(V,V)$. We consider a bracket defined on V as follows,

$$[X_1,X_2]_Q = [QX_1,X_2] + [X_1,QX_2] - Q[X_1,X_2]\,. \tag{4.16}$$

Note that this bracket measures the deviation of Q from being a derivation of the original Lie bracket of V.

We have

Proposition 4.7

(4.7.a) *The bracket* $[\ ]_Q$ *defines a Lie algebra on* V *if and only if* $\delta[Q,Q]_{FN} = 0$.

(4.7.b) *In particular,* $[Q,Q]_{FN} = 0$ *if and only if* $[\ ]_Q$ *defines a Lie algebra on* V *such that*

$$Q[X_1,X_2]_Q = [QX_1,QX_2]\,. \tag{4.17}$$

We recall from [K-SM],

Definition 4.8 *An element* $Q \in Alt^1(V,V) = Hom(V,V)$ *is called a Nijenhuis operator of the Lie algebra* V *if* $[Q,Q]_{FN} = 0$.

Proposition 4.7 shows that Nijenhuis operators induce deformations of the Lie algebra V (see [NR2]). They are sometimes called the *Nijenhuis deformation* of V. They play an important role in the Poisson-Nijenhuis structure theory ([K-SM], see also [MM]).

The proof of Proposition 4.7 is easy. In fact, (4.7.a) follows from $\delta[Q,Q]_{\mathrm{FN}} = [\delta Q_1, \delta Q_2]_{\mathrm{NR}}$; (4.7.b) holds since we have from (4.2),

$$\begin{aligned} & [Q,Q]_{\mathrm{FN}}(X_1,X_2) \\ = \;& 2([Q(X_1),Q(X_2)] - Q([QX_1,X_2] + [X_1,QX_2] - Q[X_1,X_2]))\,. \end{aligned} \tag{4.18}$$

Applying Theorem 4.5, we have

Corollary 4.9 *Let* N *be a Nijenhuis operator of the Lie algebra* V. *Then the Frölicher-Nijenhuis bracket* $[\ ]'$ *for the Lie algebra defined on* V *by the bracket*

$$[X_1,X_2]_N = [NX_1,X_2] + [X_1,NX_2] - N[X_1,X_2]\,. \tag{4.19}$$

satisfies

$$[Q_1,Q_2]' = [N,[Q_1,Q_2]_{FN}]_{NR} - [[N,Q_1]_{NR},Q_2]_{FN} - [Q_1,[N,Q_2]_{NR}]_{FN}. \tag{4.20}$$

Proof. It is clear

$$[X_1, X_2] = \delta N(X_1, X_2).$$

Therefore, we have

$$[Q_1, Q_2]' = [Q_1, Q_2]_{FN(-\delta N)} = -[Q_1, Q_2]_{FN([-\theta, N]_{NR})}.$$

The result is then a direct consequence of Theorem 4.5 and $[\ ]_{FN} = [\ ]_{FN(-\theta)}$. □.

This corollary was first proved in [N3].

Chapter 5

The Kodaira-Spencer Algebra

The Kodaira-Spencer algebra in the geometric context (cf.[KS]) came to existence several decades ago, we will consider in this chapter its algebraic version and show that such a version provides R-matrices a graded Lie algebra background. Our main contribution here is developing a rigorous approach to its construction and providing a second example of knit product structures from this graded Lie algebra.

5.1 Construction of the Kodaira-Spencer Algebra

Let V be a Lie algebra. We refer to Chapter 3 for the definition of the graded Lie algebra $SP(V)$ and that of the operator Θ. Consider the graded vector space embedding

$$\begin{aligned} \iota : Alt(V,V) &\longrightarrow SP(V), \\ \iota(Q) &= (\Theta Q, Q) . \end{aligned} \tag{5.1}$$

Lemma 5.1 *The image of ι is a subalgebra of $SP(V)$.*
Proof. For $Q_i \in Alt^{k_i}(V,V), i = 1,2$, we have

$$\begin{aligned} & [(\Theta Q_1, Q_1), (\Theta Q_2, Q_2)]_{\mathrm{SP}} \\ = \ & ([\Theta Q_1, \Theta Q_2]_{\mathrm{NR}}, [Q_1, Q_2]_{\mathrm{C}} + Q_2\Theta Q_1 - (-1)^{k_1k_2}Q_1\Theta Q_2) . \end{aligned}$$

Therefore it only needs to be proven that

$$\Theta([Q_1, Q_2]_{\mathrm{C}} + Q_2\Theta Q_1 - (-1)^{k_1k_2}Q_1\Theta Q_2) = [\Theta Q_1, \Theta Q_2]_{\mathrm{NR}} . \tag{5.2}$$

We use Lemma 3.5 and Proposition 3.8,

$$\begin{aligned}
& \Theta([Q_1,Q_2]_{\mathrm{C}}+Q_2\Theta Q_1-(-1)^{k_1k_2}Q_1\Theta Q_2) \\
= \; & (-1)^{k_2}[\Theta Q_1,Q_2]_{\mathrm{C}}+[Q_1,\Theta Q_2]_{\mathrm{C}}+\Theta(Q_2\Theta Q_1)-(-1)^{k_1k_2}\Theta(Q_1\Theta Q_2) \\
= \; & (\Theta Q_2)(\Theta Q_1)-\Theta(Q_2\Theta Q_1)+(-1)^{k_1k_2}\Theta(Q_1(\Theta Q_2))- \\
& (-1)^{k_1k_2}(\Theta Q_1)(\Theta Q_2)+\Theta(Q_2\Theta Q_1)-(-1)^{k_1k_2}\Theta(Q_1(\Theta Q_2)) \\
= \; & \Theta Q_2\Theta Q_1-(-1)^{k_1k_2}\Theta Q_1\Theta Q_2 \\
= \; & [\Theta Q_1,\Theta Q_2]_{\mathrm{NR}}\,.\square
\end{aligned}$$

We can reformulate this lemma as

Theorem 5.2 *The graded vector space $Alt(V,V)$ with bracket determined by*

$$[Q_1,Q_2]_{KS}=[Q_1,Q_2]_C+Q_2\Theta Q_1-(-1)^{k_1k_2}Q_1\Theta Q_2 \tag{5.3}$$

is a graded Lie algebra, and ι of (5.1) is an injective graded Lie algebra homomorphism.

In terms of bracket (5.3), the identity (5.2) is

$$\Theta[Q_1,Q_2]_{\mathrm{KS}}=[\Theta Q_1,\Theta Q_2]_{\mathrm{NR}}\,. \tag{5.4}$$

By (5.3) and the definition of Θ, we have

$$\begin{aligned}
& [Q_1,Q_2]_{\mathrm{KS}}(X_1,\cdots,X_{k_1+k_2}) \\
= \; & \sum_{\sigma\in sh(k_1,k_2)}(-1)^\sigma[Q_1(X_{\sigma^1}),Q_2(X_{\sigma^2}] \\
& -\sum_{\sigma\in sh(k_1,1,k_2-1)}(-1)^\sigma Q_2([Q_1(X_{\sigma^1}),X_{\sigma(k_1+1)}],X_{\sigma^3}) \\
& +(-1)^{k_1k_2}\sum_{\sigma\in sh(k_2,1,k_1-1)}(-1)^\sigma Q_1([Q_2(X_{\sigma^1}),X_{\sigma(k_2+1)}],X_{\sigma^3})\,.
\end{aligned} \tag{5.5}$$

In the geometric context, formula (5.5) defines the Kodaira-Spencer algebra([KS], cf. also [BM]). For this reason we will call the graded Lie algebra in Theorem 5.2 the *Kodaira-Spencer algebra* for the Lie algebra V and denote it by $Alt_{\mathrm{KS}}(V,V)$.

Consider the isomorphism

$$Alt(V,V)=\bigwedge V^*\bigotimes V\,.$$

Let $Q_i = v_i \bigotimes X_i, i = 1, 2$. Then we have directly from (5.5),

$$\begin{aligned} & [v_1 \bigotimes X_1, v_2 \bigotimes X_2]_{\mathrm{KS}} \\ = \; & v_1 \bigwedge v_2 \bigotimes [X_1, X_2] + v_1 \bigwedge ad^*_{X_1} v_2 \bigotimes X_2 - ad^*_{X_2} v_1 \bigwedge v_2 \bigotimes X_1 \,. \end{aligned} \tag{5.6}$$

This is another formula for the graded Lie bracket of the Kodaira-Spencer algebra besides (5.3) and (5.5). We remark the fact that (5.6) defines a graded Lie algebra on $Alt(V, V)$ can also be proved directly through a calculation with the help of the following two obvious identities:

$$ad^*_X ad^*_Y - ad^*_Y ad^*_X = ad^*_{[X,Y]} \,, \tag{5.7}$$

$$ad^*_X (v_1 \bigwedge v_2) = ad^*_X v_1 \bigwedge v_2 + v_1 \bigwedge ad^*_X v_2 \,. \tag{5.8}$$

From (5.5), we especially have

Corollary 5.3 *For $Q_1, Q_2 \in Alt^1(V, V)$, there holds*

$$\begin{aligned} & [Q_1, Q_2]_{KS}(X_1, X_2) \\ = \; & [Q_1(X_1), Q_2(X_2)] + [Q_2(X_1), Q_1(X_2)] \\ & - Q_1([Q_2(X_1), X_2] + [X_1, Q_2(X_2)]) \\ & - Q_2([Q_1(X_1), X_2] + [X_1, Q_1(X_2)]) \,. \end{aligned} \tag{5.9}$$

We now show how the Kodaira-Spencer algebra is related to R-matrices in the sense of Semenov-Tian-Shansky.

For $Q \in Alt^1(V, V) = Hom(V, V)$, we denote $T(Q) \in Alt^2(V, V)$ for

$$T(Q) = \frac{1}{2}[Q, Q]_{\mathrm{KS}} \,. \tag{5.10}$$

Proposition 5.4 *The bracket*

$$[X_1, X_2]_Q = [QX_1, X_2] + [X_1, QX_2] \tag{5.11}$$

defines a Lie algebra on V if and only if

$$\Theta T(Q) = 0 \,. \tag{5.12}$$

In particular, this condition is satisfied when $T(Q) = c\theta$ with c a constant real number.

Since

$$[QX_1, X_2] + [X_1, QX_2] = \theta Q(X_1, X_2) = -\Theta Q(X_1, X_2),$$

this proposition is a direct consequence of (5.4).

Note that $T(Q) = 0$ and $T(Q) = -\theta$ are respectively equivalent to

$$[QX_1, QX_2] - [QX_1, X_2] - [X_1, QX_2] = 0$$

and

$$[QX_1, QX_2] - [QX_1, X_2] - [X_1, QX_2] + [X_1, X_2] = 0.$$

They are exactly the classical Yang-Baxter equation and the modified classical Yang-Baxter equation, and their solutions are called *R-matrices* by Semenov-Tian-Shansky ([STS]).

To end this section, let us prove that the Kodaira-Spencer algebra is also a differential graded Lie algebra.

Theorem 5.5 *For $Q_i \in Alt^{k_i}(V, V), i = 1, 2$, we have*

$$D[Q_1, Q_2]_{KS} = [DQ_1, Q_2]_{KS} + (-1)^{k_1}[Q_1, DQ_2]_{KS}. \tag{5.13}$$

Proof. We use (5.6) and the fact

$$d\, ad^*_X = ad^*_X d.$$

Without loss of generality, we suppose $Q_i = v_i \bigotimes X_i, i = 1, 2$.

$$\begin{aligned}
& D[v_1 \bigotimes X_1, v_2 \bigotimes X_2]_{\mathrm{KS}} \\
= \;& D(v_1 \bigwedge v_2 \bigotimes [X_1, X_2] + v_1 \bigwedge ad^*_{X_1} v_2 \bigotimes X_2 - ad^*_{X_2} v_1 \bigwedge v_2 \bigotimes X_1) \\
= \;& d(v_1 \bigwedge v_2) \bigotimes [X_1, X_2] + d(v_1 \bigwedge ad^*_{X_1} v_2) \bigotimes X_2 - d(ad^*_{X_2} v_1 \bigwedge v_2) \bigotimes X_1 \\
= \;& dv_1 \bigwedge v_2 \bigotimes [X_1, X_2] + dv_1 \bigwedge ad^*_{X_1} v_2 \bigotimes X_2 - ad^*_{X_2} dv_1 \bigwedge v_2 \bigotimes X_1 \\
& +(-1)^{k_1}(v_1 \bigwedge dv_2 \bigotimes [X_1, X_2] + v_1 \bigwedge ad^*_{X_1} dv_2 \bigotimes X_2 - ad^*_{X_2} v_1 \bigwedge dv_2 \bigotimes X_1) \\
= \;& [D(v_1 \bigotimes X_1), (v_2 \bigotimes X_2)]_{\mathrm{KS}} + (-1)^{k_1}[(v_1 \bigotimes X_1), D(v_2 \bigotimes X_2)]_{\mathrm{KS}}.
\end{aligned}$$

This proves (5.13). □

5.2 A Second Knit Product

The Kodaira-Spencer algebra provides us a second example of knit product structures. We explore this construction in this section.

Let $(P_i, Q_i) \in SP^{k_i}(V), i = 1, 2$ and define

$$
\begin{aligned}
& [(P_1, Q_1), (P_2, Q_2)]'_{\mathrm{KP}} \\
= \; & ([P_1, P_2]_{\mathrm{NR}} + ([Q_2, P_1]_{\mathrm{C}} - P_1(\Theta Q_2)) - (-1)^{k_1 k_2}([Q_1, P_2]_{\mathrm{C}} - P_2(\Theta Q_1)), \\
& [Q_1, Q_2]_{\mathrm{KS}} + Q_2 P_1 - (-1)^{k-1k_2} Q_1 P_2),
\end{aligned}
\tag{5.14}
$$

we have

Theorem 5.6 *On the graded vector space $Alt(V, V)[1] \bigoplus Alt(V, V)$, the bracket determined by (5.14) defines a graded Lie algebra structure.*

This new graded Lie algebra structure is the knit product of the Nijenhuis-Richardson algebra and the Kodaira-Spencer algebra, because its restrictions to the first and the second factors are respectively these algebras.

Proof. Note that

$$
\begin{aligned}
& [(P_1 + \Theta Q_1, Q_1), (P_2 + \Theta Q_2, Q_2)]_{\mathrm{SP}} \\
= \; & ([P_1 + \Theta Q_1, P_2 + \Theta Q_2]_{\mathrm{NR}}, [Q_1, Q_2]_{\mathrm{C}} + Q_2(P_1 + \Theta Q_1) - (-1)^{k_1 k_2} Q_1(P_2 + \Theta Q_2)) \\
= \; & ([P_1, P_2]_{\mathrm{NR}} + [P_1, \Theta Q_2]_{\mathrm{NR}} + [\Theta Q_1, P_2]_{\mathrm{NR}} + \Theta[Q_1, Q_2]_{\mathrm{KS}}, \\
& [Q_1, Q_2]_{\mathrm{KS}} + Q_2 P_1 - (-1)^{k_1 k_2} Q_1 P_2) \\
= \; & ([P_1, P_2]_{\mathrm{NR}} + ([P_1, \Theta Q_2]_{\mathrm{NR}} - \Theta(Q_2 P_1)) - (-1)^{k_1 k_2}([P_2, \Theta Q_1]_{\mathrm{NR}} - \\
& \Theta(Q_1 P_2)) + \Theta([Q_1, Q_2]_{\mathrm{KS}} + Q_2 P_1 - (-1)^{k_1 k_2} Q_1 P_2), \\
& [Q_1, Q_2]_{\mathrm{KS}} + Q_2 P_1 - (-1)^{k_1 k_2} Q_1 P_2).
\end{aligned}
$$

Applying (2.12) and Lemma 3.5, we have

$$[P_1, \Theta Q_2]_{\mathrm{NR}} - \Theta(Q_2 P_1) = [Q_2, P_1]_{\mathrm{C}} - P_1(\Theta Q_2)$$

and

$$[P_2, \Theta Q_1]_{\mathrm{NR}} - \Theta(Q_1 P_2) = [Q_1, P_2]_{\mathrm{C}} - P_2(\Theta Q_1).$$

Therefore,

$$\begin{aligned}
&[(P_1 + \Theta Q_1, Q_1), (P_2 + \Theta Q_2, Q_2)]_{\mathrm{SP}} \\
= \; &([P_1, P_2]_{\mathrm{NR}} + ([Q_2, P_1]_{\mathrm{C}} - P_1(\Theta Q_2)) - (-1)^{k_1 k_2}([Q_1, P_2]_{\mathrm{C}} - P_2(\Theta Q_1)) + \\
&\Theta([Q_1, Q_2]_{\mathrm{KS}} + Q_2 P_1 - (-1)^{k_1 k_2} Q_1 P_2), \\
&[Q_1, Q_2]_{\mathrm{KS}} + Q_2 P_1 - (-1)^{k_1 k_2} Q_1 P_2).
\end{aligned} \tag{5.15}$$

If we define

$$\begin{aligned}
\hat{\iota} : Alt(V, V)[1] \bigoplus Alt(V, V) &\longrightarrow SP(V), \\
\hat{\iota}(P, Q) &= (P + \Theta Q, Q),
\end{aligned} \tag{5.16}$$

it follows from (5.15),

$$\begin{aligned}
&\hat{\iota}[(P_1, Q_1), (P_2, Q_2)]'_{\mathrm{KP}} \\
= \; &[\hat{\iota}(P_1, Q_1), \hat{\iota}(P_2, Q_2)]_{\mathrm{SP}}.
\end{aligned} \tag{5.17}$$

The theorem is implied in this identity since $\hat{\iota}$ is a graded vector space isomorphism. □

From the proof, it is obvious that $\hat{\iota}$ of (5.16) is a graded Lie algebra isomorphism from the knit product structure of Theorem 5.6 to the semi-direct product structure on $SP(V)$.

By the expression (5.14), we know the derivatively knitted pair of representations corresponding to this knit product structure are

$$\begin{aligned}
&\alpha' : Alt(V, V)[1] \longrightarrow End(Alt(V, V)), \\
&\alpha'(P)Q = QP \\
&\beta' : Alt_{\mathrm{KS}}(V, V) \longrightarrow End(Alt(V, V)[1]), \\
&\beta'(Q)P = (ad_Q + \Im(\Theta Q))P.
\end{aligned} \tag{5.18}$$

Here,

$$\begin{aligned} ad_Q P &= [Q,P]_{\mathrm{C}}\,, \\ \Im(\Theta Q)P &= P(\Theta Q)\,. \end{aligned}$$

The fact α' is a graded Lie algebra homomorphism is already known in Chapter 3 (cf. Proposition 3.1.a). We can directly prove that β' is also a graded Lie algebra homomorphism as follows.

First, we have from the graded Jacobi identity of the cup algebra $Alt_{\mathrm{C}}(V,V)$,

$$ad_{[Q_1,Q_2]_{\mathrm{C}}} = [ad_{Q_1}, ad_{Q_2}]\,.$$

We also proved (cf. (3.3))

$$[\Im(\Theta Q_1), \Im(\Theta Q_2)] = \Im([\Theta Q_1, \Theta Q_2]_{\mathrm{NR}})\,.$$

Therefore,

$$\begin{aligned} & [\beta'(Q_1), \beta'(Q_2)] \\ = \; & [ad_{Q_1} + \Im(\Theta Q_1), ad_{Q_2} + \Im(\Theta Q_2)] \\ = \; & ad_{[Q_1,Q_2]_{\mathrm{C}}} + \Im([\Theta Q_1, \Theta Q_2]_{\mathrm{NR}}) + [ad_{Q_1}, \Im(\Theta Q_2)] + \\ & [\Im(\Theta Q_1), ad_{Q_2}] \\ = \; & ad_{[Q_1,Q_2]_{\mathrm{C}}} + \Im(\Theta[Q_1,Q_2]_{\mathrm{KS}}) + [ad_{Q_1}, \Im(\Theta Q_2)] + \\ & [\Im(\Theta Q_1), ad_{Q_2}]\,. \end{aligned}$$

In order to prove

$$\beta'[Q_1,Q_2]_{\mathrm{KS}} = [\beta'(Q_1), \beta'(Q_2)]\,,$$

we only need to show

$$[\Im(P), ad_Q] = ad_{QP}\,.$$

However, writing this identity explicitly, we can see that it is equivalent to Proposition 3.1.a.

We remark that formulae analogous to (2.6) can be written down for this knit product structure.

To end this section, we notice that the knit product structure in Theorem 5.6 enables us to consider the deformation of Lie algebra V through a pair of operators rather than just a R-matrix.

Proposition 5.7 *Let $P \in Alt^2(V,V), Q \in Alt^1(V,V)$. Then*

$$[(P,Q),(P,Q)]'_{KP} = 0$$

if and only if the following two conditions hold,

(5.7.a) *the bracket determined by*

$$[X_1, X_2]'_{(P,Q)} = [QX_1, X_2] + [X_1, QX_2] - P(X_1, X_2)$$

defines a Lie algebra structure on V.

(5.7.b) *We have*

$$Q[X_1, X_2]'_{(P,Q)} = [QX_1, QX_2].$$

We hope this proposition is useful in integrable Hamiltonian system theory.

Chapter 6

The Gelfand-Dorfman Algebra

In this chapter we construct the Gelfand-Dorfman algebra and establish its relation the classical r-matrices and the algebraic Schouten-Nijenhuis algebra.

6.1 Construction of the Gelfand-Dorfman Algebra

In order to construct the Gelfand-Dorfman algebra, we recall that the adjoint representation of a Lie algebra V on itself induces representations on all the vector spaces $\bigwedge^k V$, $k = 1, 2, \cdots$. Using the same notation *"ad"* for all these representations, we have for any $X, X_i \in V$, $T_i \in \bigwedge^{k_i} V, i = 1, 2,$

$$ad_{X_1} ad_{X_2} - ad_{X_2} ad_{X_1} = ad_{[X_1, X_2]}, \tag{6.1}$$

$$ad_X(T_1 \bigwedge T_2) = (ad_X T_1) \bigwedge T_2 + T_1 \bigwedge (ad_X T_2). \tag{6.2}$$

Consider the graded vector space

$$\bigwedge V \bigotimes V = \bigoplus_{k \geq 0} (\bigwedge^k V \bigotimes V).$$

Theorem 6.1 *The bracket determined by*

$$\begin{aligned} [T_1 \bigotimes X_1, T_2 \bigotimes X_2]_{GD} &= T_1 \bigwedge T_2 \bigotimes [X_1, X_2] + T_1 \bigwedge ad_{X_1} T_2 \bigotimes X_2 - \\ & \quad ad_{X_2} T_1 \bigwedge T_2 \bigotimes X_1 \end{aligned} \tag{6.3}$$

defines a graded Lie algebra on $\bigwedge V \bigotimes V$.

Proof. We only have to consider simple tensors. We have

$$\begin{aligned}
& [T_2 \bigotimes X_2, T_1 \bigotimes X_1]_{\mathrm{GD}} \\
= \; & T_2 \bigwedge T_1 \bigotimes [X_2, X_1] + T_2 \bigwedge ad_{X_2} T_1 \bigotimes X_1 - ad_{X_1} T_2 \bigwedge T_1 \bigotimes X_2 \\
= \; & -(-1)^{k_1 k_2} (T_1 \bigwedge T_2 \bigotimes [X_1, X_2] + T_1 \bigwedge ad_{X_1} T_2 \bigotimes X_2 - ad_{X_2} T_1 \bigwedge T_2 \bigotimes X_1) \\
= \; & -(-1)^{k_1 k_2} [T_1 \bigotimes X_1, T_2 \bigotimes X_2]_{\mathrm{GD}}.
\end{aligned}$$

This proves the graded anti-commutativity.

The graded Jacobi identity for the bracket (6.3) follows from the Jacobi identity for the Lie algebra V and (6.1) and (6.2).□

I was led to the graded Lie algebra in the above theorem when I was studying the Gelfand and Dorfman work on the integrability of Dirac structures ([D]). As we will see later, the bracket of this graded Lie algebra provides us an expression of the bracket of the algebraic Schouten-Nijenhuis algebra in terms of alternating mappings. The simplest form of such an expression, i.e., the bracket of two degree 1 elements of the algebraic Schouten-Nijenhuis algebra considered as alternating mappings, was first given by Gelfand and Dorfman. For this reason, we will call this graded Lie algebra the *Gelfand-Dorfman algebra* associated with Lie algebra V.

Several observations are ready to be made here.

First,the construction in Theorem 6.1 generalizes easily to $\bigwedge V \otimes W$ where W is also a Lie algebra. In this case, the adjoint representation should be replaced by a representation of W on V. This observation was pointed out to me by Professor Stasheff.

Second, the Gelfand-Dorfman algebra provides a graded Lie algebra background for general (not necessarily anti-symmetric) *r-matrices* ([Drl]). Actually, comparing the equation defining r-matrices in [Drl] with (6.3), we can see that r-matrices are nothing but degree 1, bracket-square 0 elements of the Gelfand-Dorfman algebra.

Third, the Kodaira-Spencer algebra and the Gelfand-Dorfman algebra become identical when Lie algebra V is semisimple. In this case, there is a non-degenerate invariant bilinear form on V through which we can identify V and V^* and further identify the adjoint and coadjoint representation of this Lie algebra. This observation is then clear from (5.6) and (6.3).

6.2 The Cyclic Subalgebra

Note that

$$Alt^k(V^*, V) = \bigwedge^k V \bigotimes V, \quad k = 1, 2, \cdots.$$

Therefore, we have a graded vector space isomorphism

$$Alt(V^*, V) = \bigwedge V \bigotimes V \tag{6.4}$$

with

$$Alt(V^*, V) = \bigoplus_{k \geq 0} Alt^k(V^*, V).$$

Through this isomorphism we can also think of the Gelfand-Dorfman algebra as defined on $Alt(V^*, V)$. Let us express its graded Lie bracket in terms of this graded vector space first.

Recall the coadjoint representation of a Lie algebra V. It is given for $X_1, X_2 \in V$, $\psi \in V^*$ by

$$< ad^*_{X_1} \psi, X_2 >=< \psi, [X_2, X_1] > . \tag{6.5}$$

Hence, there holds

$$ad^*_{X_1} \psi = -\psi(ad_{X_1})$$

as operators on V. We can rewrite this as

$$< ad_{X_1} X_2, \psi >=< -X_2, ad^*_{X_1} \psi > . \tag{6.6}$$

The following lemma generalizes (6.6).

Lemma 6.2 *For $T \in \bigwedge^k V, \psi_1, \cdots, \psi_k \in V^*$ and $X \in V$, we have*

$$< ad_X T, \psi_1 \bigwedge \cdots \bigwedge \psi_k >= -\sum_{i=1}^{k} < T, \psi_1 \bigwedge \cdots \bigwedge ad^*_X \psi_i \bigwedge \cdots \bigwedge \psi_k). \tag{6.7}$$

Proof. Without loss of generality, we assume

$$T = X_1 \bigwedge \cdots \bigwedge X_k.$$

Then

$$ad_X T = \sum_{i=1}^{k} X_1 \bigwedge \cdots \bigwedge ad_X X_i \bigwedge \cdots X_k.$$

By the pairing

$$< X_1 \bigwedge \cdots \bigwedge X_k, \psi_1 \bigwedge \cdots \bigwedge \psi_k >= det(X_i(\psi_j)),$$

we have

$$\begin{aligned}
& < ad_X T, \psi_1 \bigwedge \cdots \bigwedge \psi_k > \\
= \ & \sum_{i=1}^{k} < X_1 \bigwedge \cdots \bigwedge ad_X X_i \bigwedge \cdots \bigwedge X_k, \psi_1 \bigwedge \cdots \bigwedge \psi_k > \\
= \ & \sum_{i=1}^{k} \sum_{\sigma \in \Sigma_k} (-1)^\sigma < X_1, \psi_{\sigma(1)} > \cdots < ad_X X_i, \psi_{\sigma(i)} >< \cdots X_k, \psi_{\sigma(k)} > \\
= \ & - \sum_{i=1}^{k} \sum_{\sigma \in \Sigma_k} (-1)^\sigma < X_1, \psi_{\sigma(1)} > \cdots < X_i, ad_X^* \psi_{\sigma(i)} > \cdots < X_k, \psi_{\sigma(k)} > \\
= \ & - \sum_{i=1}^{k} < X_1 \bigwedge \cdots \bigwedge X_k, \psi_1 \bigwedge \cdots \bigwedge ad_X^* \psi_i \bigwedge \cdots \bigwedge \psi_k > \\
= \ & - \sum_{i=1}^{k} < T, \psi_1 \bigwedge \cdots \bigwedge ad_X^* \psi_i \bigwedge \cdots \bigwedge \psi_k > .
\end{aligned}$$

This completes the proof. □

Note that

$$\begin{aligned}
& \sum_{i=1}^{k} < T, \psi_1 \bigwedge \cdots \bigwedge ad_X^* \psi_i \bigwedge \cdots \bigwedge \psi_k > \\
= \ & \sum_{i=1}^{k} (-1)^{i-1} < T, ad_X^* \psi_i \bigwedge \psi_1 \bigwedge \cdots \bigwedge \widehat{\psi_i} \bigwedge \cdots \bigwedge \psi_k > .
\end{aligned} \tag{6.8}$$

A direct application of this identity and Lemma 6.2 gives

Theorem 6.3 *Let $I_i \in Alt^{k_i}(V^*, V), i = 1, 2$. We have*

$$\begin{aligned}
& [I_1, I_2]_{GD}(\psi_1, \cdots, \psi_{k_1+k_2}) \\
= & \sum_{\sigma \in sh(k_1,k_2)} (-1)^\sigma [I_1(\psi_{\sigma^1}), I_2(\psi_{\sigma^2})] \\
& - \sum_{\sigma \in sh(k_1,1,k_2-1)} (-1)^\sigma I_2(ad^*_{I_1(\psi_{\sigma^1})} \psi_{\sigma(k_1+1)}, \psi_{\sigma^3}) \\
& +(-1)^{k_1 k_2} \sum_{\sigma \in sh(k_2,1,k_1-1)} (-1)^\sigma I_1(ad^*_{I_2(\psi_{\sigma^1})} \psi_{\sigma(k_2+1)}, \psi_{\sigma^3})
\end{aligned} \tag{6.9}$$

Now, we use this theorem to establish an interesting subalgebra of the Gelfand-Dorfman algebra $Alt(V^*, V) = \bigwedge V \bigotimes V$. This is our main goal for this section.

Definition 6.4 *Let $I \in Alt^k(V^*, V)$. I is called cyclic if for all $\psi_1, \cdots, \psi_{k+1} \in V^*$,*

$$< \psi_1, I(\psi_2, \cdots, \psi_{k+1}) >= (-1)^k < I(\psi_1, \cdots, \psi_k), \psi_{k+1} > . \tag{6.10}$$

Since I is alternating, (6.10) is equivalent to

$$\begin{aligned}
& < \psi_1, I(\psi_2, \cdots, \psi_{k+1}) > \\
= & (-1)^{i-1} < \psi_i, I(\psi_1, \cdots, \widehat{\psi_i}, \cdots, \psi_{k+1}) >
\end{aligned} \tag{6.11}$$

for arbitrary $i = 1, 2, \cdots, k + 1$.

We denote $cAlt^k(V^*, V)$ for all the cyclic elements in $Alt^k(V^*, V)$ and define the graded vector space

$$cAlt(V^*, V) = \bigoplus_{k \geq 0} cAlt^k(V^*, V).$$

Theorem 6.5 *If I_1, I_2 are cyclic, so is $[I_1, I_2]_{GD}$. Therefore, $cAlt^k(V^*, V)$ is a subalgebra of $Alt(V^*, V)$.*

Proof. The following calculations use extensively (6.6) and the cyclic property of I_1 and I_2.

$$\begin{aligned}
&\sum_{\sigma\in sh(k_1,k_2)} (-1)^\sigma < [I_1(\psi_{\sigma^1}), I_2(\psi_{\sigma^2})], \psi_{k_1+k_2+1} > \\
= &\sum_{\sigma\in sh_1(k_1,k_2)} (-1)^\sigma < [I_1(\psi_1, \psi_{\sigma^1}), I_2(\psi_{\sigma^2})], \psi_{k_1+k_2+1} > \\
&+ \sum_{\sigma\in sh_2(k_1,k_2)} (-1)^\sigma < [I_1(\psi_{\sigma^1}), I_2(\psi_1, \psi_{\sigma^2})], \psi_{k_1+k_2+1} > \\
= &\sum_{\sigma\in sh_1(k_1,k_2)} (-1)^\sigma < I_1(\psi_1, \psi_{\sigma^1}), ad^*_{I_2(\psi_{\sigma^2})}\psi_{k_1+k_2+1} > \\
&- \sum_{\sigma\in sh_2(k_1,k_2)} (-1)^\sigma < I_2(\psi_1, \psi_{\sigma^2}), ad^*_{I_1(\psi_{\sigma^1})}\psi_{k_1+k_2+1} > \\
= &- \sum_{\sigma\in sh_1(k_1,k_2)} (-1)^\sigma < \psi_1, I_1(ad^*_{I_2(\psi_{\sigma^2})}\psi_{k_1+k_2+1}, \psi_{\sigma^1}) > \\
&+ \sum_{\sigma\in sh_2(k_1,k_2)} (-1)^\sigma < \psi_1, I_2(ad^*_{I_1(\psi_{\sigma^1})}\psi_{k_1+k_2+1}, \psi_{\sigma^2} > .
\end{aligned}$$

$$\begin{aligned}
&\sum_{\sigma\in sh(k_1,1,k_2-1)} (-1)^\sigma < I_2(ad^*_{I_1(\psi_{\sigma^1})}\psi_{\sigma(k_1+1)}, \psi_{\sigma^3}), \psi_{k_1+k_2+1} > \\
= &\sum_{\sigma\in sh_1(k_1,1,k_2-1)} (-1)^\sigma(-1)^{k_2} < ad^*_{I_1(\psi_1,\psi_{\sigma^2})}\psi_{\sigma(k_1+1)}, I_2(\psi_{\sigma^3}, \psi_{k_1+k_2+1}) > \\
&+ \sum_{\sigma\in sh_2(k_1,1,k_2)} (-1)^\sigma(-1)^{k_2} < ad^*_{I_1(\psi_{\sigma^1})}\psi_1, I_2(\psi_{\sigma^3}, \psi_{k_1+k_2+1}) > \\
&- \sum_{\sigma\in sh_3(k_1,1,k_2-1)} (-1)^\sigma(-1)^{k_2} < \psi_1, I_2(ad^*_{I_1(\psi_{\sigma^1})}\psi_{\sigma(k_1+1)}, \psi_{\sigma^3}, \psi_{k_1+k_2+1}) > \\
= &- \sum_{\sigma\in sh_1(k_1,1,k_2-1)} (-1)^\sigma(-1)^{k_2} < I_1(\psi_1, \psi_{\sigma^1}), ad^*_{I_2(\psi_{\sigma^3},\psi_{k_1+k_2+1})}\psi_{\sigma(k_1+1)} > \\
&- \sum_{\sigma\in sh_2(k_1,1,k_2-1)} (-1)^\sigma(-1)^{k_2} < \psi_1, [I_2(\psi_{\sigma^3}, \psi_{k_1+k_2+1}), I_1(\psi_{\sigma^1})] > \\
&- \sum_{\sigma\in sh_3(k_1,1,k_2-1)} (-1)^\sigma(-1)^{k_2} < \psi_1, I_2(ad^*_{I_1(\psi_{\sigma^1})}\psi_{\sigma(k_1+1)}, \psi_{\sigma^3}, \psi_{k_1+k_2+1}) > \\
= &\sum_{\sigma\in sh_1(k_1,1,k_2-1)} (-1)^\sigma(-1)^{k_2} < \psi_1, I_1(ad^*_{I_2(\psi_{\sigma^3},\psi_{k_1+k_2+1})}\psi_{\sigma(k_1+1)}, \psi_{\sigma^1}) > \\
&- \sum_{\sigma\in sh_2(k_1,1,k_2-1)} (-1)^\sigma(-1)^{k_2} < \psi_1, [I_1(\psi_{\sigma^1}), I_2(\psi_{\sigma^3}, \psi_{k_1+k_2+1})] >
\end{aligned}$$

$$- \sum_{\sigma \in sh_3(k_1,1,k_2-1)} (-1)^{\sigma}(-1)^{k_2} < \psi_1, I_2(ad^*_{I_1(\psi_{\sigma^1})}\psi_{\sigma(k_1+1)}, \psi_{\sigma^3}, \psi_{k_1+k_2+1}) > .$$

Similarly, we have

$$\begin{aligned}
& \sum_{\sigma \in sh(k_2,1,k_1-1)} (-1)^{\sigma} < I_1(ad^*_{I_2(\psi_{\sigma^1})}\psi_{\sigma(k_2+1)}, \psi_{\sigma^3}), \psi_{k_1+k_2+1} > \\
= & \sum_{\sigma \in sh_1(k_2,1,k_1-1)} (-1)^{\sigma}(-1)^{k_1} < \psi_1, I_2(ad^*_{I_1(\psi_{\sigma^3},\psi_{k_1+k_2+1})}\psi_{\sigma(k_2+1)}, \psi_{\sigma^1}) > \\
& - \sum_{\sigma \in sh_2(k_2,1,k_1-1)} (-1)^{\sigma}(-1)^{k_1} < \psi_1, [I_2(\psi_{\sigma^1}), I_1(\psi_{\sigma^3}, \psi_{k_1+k_2+1})] > \\
& - \sum_{\sigma \in sh_3(k_2,1,k_1-1)} (-1)^{\sigma}(-1)^{k_1} < \psi_1, I_1(ad^*_{I_2(\psi_{\sigma^1})}\psi_{\sigma(k_2+1)}, \psi_{\sigma^3}, \psi_{k_1+k_2+1}) > .
\end{aligned}$$

By the above three identities, we can evaluate

$$< [I_1, I_2]_{\mathrm{GD}}(\psi_1, \cdots, \psi_{k_1+k_2}), \psi_{k_1+k_2+1} > .$$

Note that,

$$\begin{aligned}
& \sum_{\sigma \in sh_2(k_1,1,k_2-1)} (-1)^{\sigma}(-1)^{k_2} < \psi_1, [I_1(\psi_{\sigma^1}), I_2(\psi_{\sigma^3}, \psi_{k_1+k_2+1})] > \\
& -(-1)^{k_1 k_2} \sum_{\sigma \in sh_2(k_2,1,k_1-1)} (-1)^{\sigma}(-1)^{k_1} < \psi_1, [I_2(\psi_{\sigma^1}), I_1(\psi_{\sigma^3}, \psi_{k_1+k_2+1})] > \\
= & (-1)^{k_1+k_2} \sum_{\sigma \in sh(k_1,k_2)} < \psi_1, [I_1(\psi_{\sigma^1}), I_2(\psi_{\sigma^2})] > .
\end{aligned}$$

Considering the position of $\psi_{k_1+k_2+1}$, we can also show,

$$\begin{aligned}
& \sum_{\sigma \in sh_2(k_1,k_2)} (-1)^{\sigma} < \psi_1, I_2(ad^*_{I_1(\psi_{\sigma^1})}\psi_{k_1+k_2+1}, \psi_{\sigma^2}) > \\
& + \sum_{\sigma \in sh_3(k_1,1,k_2-1)} (-1)^{\sigma}(-1)^{k_2} < \psi_1, I_2(ad^*_{I_1(\psi_{\sigma^1})}\psi_{\sigma(k_1+1)}, \psi_{\sigma^3}, \psi_{k_1+k_2+1}) > \\
& +(-1)^{k_1 k_2} \sum_{\sigma \in sh_1(k_2,1,k_1-1)} (-1)^{\sigma}(-1)^{k_1} < \psi_1, I_2(ad^*_{I_1(\psi_{\sigma^3},\psi_{k_1+k_2+1})}\psi_{\sigma(k_2+1)}, \psi_{\sigma^1}) > \\
= & -(-1)^{k_1+k_2} \sum_{\sigma \in sh(k_1,1,k_2-1)} (-1)^{\sigma} < \psi_1, I_2(ad^*_{I_1(\psi_{\sigma^1})}\psi_{\sigma(k_1+2)}, \psi_{\sigma^3}) >
\end{aligned}$$

and

$$- \sum_{\sigma \in sh_1(k_1,k_2)} (-1)^{\sigma} < \psi_1, I_1(ad^*_{I_2(\psi_{\sigma^2})}\psi_{k_1+k_2+1}, \psi_{\sigma^1}) >$$

$$
\begin{aligned}
& - \sum_{\sigma \in sh_1(k_1,1,k_2-1)} (-1)^{\sigma}(-1)^{k_2} < \psi_1, I_1(ad^*_{I_2(\psi_{\sigma^3},\psi_{k_1+k_2+1})}\psi_{\sigma(k_1+1)}, \psi_{\sigma^1}) \\
& -(-1)^{k_1 k_2} \sum_{\sigma \in sh_3(k_2,1,k_1-1)} (-1)^{\sigma}(-1)^{k_1} < \psi_1, I_1(ad^*_{I_2(\psi_{\sigma^1})}\psi_{\sigma(k_2+1)}, \psi_{\sigma^3}, \psi_{k_1+k_2+1}) > \\
= \; & (-1)^{k_1+k_2}(-1)^{k_1 k_2} \sum_{\sigma \in sh(k_2,1,k_1-1)} (-1)^{\sigma} < \psi_1, I_1(ad^*_{I_2(\psi_{\sigma^1})}\psi_{\sigma(k_2+2)}, \psi_{\sigma^3}) > .
\end{aligned}
$$

Therefore, we have from (6.9),

$$
\begin{aligned}
& < [I_1, I_2]_{\mathrm{GD}}(\psi_1, \cdots, \psi_{k_1+k_2}), \psi_{k_1+k_2+1} > \\
= \; & (-1)^{k_1+k_2} < \psi_1, [I_1, I_2]_{\mathrm{GD}}(\psi_2, \cdots, \psi_{k_1+k_2+1}) > .
\end{aligned}
$$

Hence, $[I_1, I_2]_{\mathrm{GD}}$ is cyclic and the proof is completed. □

We will reveal in next section the exact meaning of the subalgebra in the above theorem.

6.3 The Schouten-Nijenhuis Algebra

In this section, we show that the subalgebra of cyclic elements of the Gelfand-Dorfman algebra we established in the last section is isomorphic to the Schouten- Nijenhuis algebra for the Lie algebra V.

We first show that $cAlt(V^*, V)$ and $\bigwedge V[1]$ are isomorphic as graded vector spaces. This is archived through explicitly constructing a pair of mutually inverse homomorphisms between them.

For $I \in cAlt^k(V^*, V)$, let

$$< I^\sharp, \psi_1 \bigwedge \cdots \bigwedge \psi_{k+1}) =< \psi_1, I(\psi_2, \cdots, \psi_{k+1}) > . \tag{6.12}$$

The cyclic property of I implies $I^\sharp \in \bigwedge^{k+1} V$. Therefore, we have a well-defined graded vector space homomorphism,

$$.^\sharp : cAlt(V^*, V) \longrightarrow \bigwedge V[1].$$

For $S \in \bigwedge^{k+1} V$, $S = X_1 \bigwedge \cdots X_{k+1}$, let

$$S^\flat(\psi_1, \cdots, \psi_k) = \sum_{i=1}^{k+1} (-1)^{i-1} < X_1 \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge \cdots \bigwedge X_{k+1}, \psi_1 \bigwedge \cdots \bigwedge \psi_k > X_i. \tag{6.13}$$

Note that

$$\begin{aligned} & < X_1 \bigwedge \cdots \bigwedge X_{k+1}, \psi_1 \bigwedge \cdots \bigwedge \psi_{k+1} > \\ = & \sum_{i=1}^{k+1} (-1)^{i+1} X_i(\psi_1) < X_1 \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge \cdots \bigwedge X_{k+1}), \psi_2 \bigwedge \cdots \bigwedge \psi_{k+1} > \\ = & \sum_{i=1}^{k+1} (-1)^{k+1+i} X_i(\psi_{k+1}) < X_1 \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge \cdots \bigwedge X_{k+1}, \psi_1 \bigwedge \cdots \bigwedge \psi_k > \\ = & (-1)^k \sum_{i=1}^{k+1} (-1)^{i+1} X_i(\psi_{k+1}) < X_1 \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge \cdots \bigwedge X_{k+1}, \psi_1 \bigwedge \cdots \bigwedge \psi_k > \end{aligned} \tag{6.14}$$

(This is nothing but two different expansions of the determinant $\det(X_i(\psi_j)))$. Therefore, we have

$$< S, \psi_1 \bigwedge \cdots \bigwedge \psi_{k+1} >$$

$$
\begin{aligned}
&= \; < \psi_1, S^\flat(\psi_2, \cdots, \psi_{k+1}) > \\
&= \; (-1)^k < S^\flat(\psi_1, \cdots, \psi_k), \psi_{k+1} > .
\end{aligned}
\tag{6.15}
$$

This implies $S^\flat \in cAlt^k(V^*, V)$. Hence we get a second well-defined graded vector space homomorphism,

$$\cdot^\flat : \bigwedge V[1] \longrightarrow cAlt(V^*, V).$$

By (6.12) and (6.15), we get

$$
\begin{aligned}
& < S^\flat)^\sharp, \psi_1 \bigwedge \cdots \bigwedge \psi_{k+1} > \\
= \; & < \psi_1, S^\flat(\psi_2, \cdots, \psi_{k+1}) > \\
= \; & < S, \psi_1 \bigwedge \cdots \bigwedge \psi_{k+1} >
\end{aligned}
$$

and

$$
\begin{aligned}
& < \psi_1, (I^\sharp)^\flat(\psi_2, \cdots, \psi_{k+1}) > \\
= \; & < I^\sharp, \psi_1 \bigwedge \cdots \bigwedge \psi_{k+1} > \\
= \; & < \psi_1, I(\psi_2, \cdots, \psi_{k+1}) > .
\end{aligned}
$$

These two identities show

$$
\begin{aligned}
(S^\flat)^\sharp &= S, \\
(I^\sharp)^\flat &= I.
\end{aligned}
$$

We have proved

Theorem 6.6 *The maps $\cdot^\sharp$ and $\cdot^\flat$ determine mutually inverse graded vector space isomorphisms between $cAlt(V^*, V)$ and $\bigwedge V[1]$.*

The following theorem is the main result of this section, which states that under $\cdot^\sharp$ and $\cdot^\flat$ the graded Lie algebra $cAlt(V^*, V)$ of Theorem 6.5 is isomorphic to the Schouten-Nijenhuis algebra.

Theorem 6.7 *For $I_i \in cAlt^{k_i}(V^*, V)$ and $S_i \in \bigwedge^{k_i+1} V, i = 1, 2,$ we have*

$$[I_1, I_2]^\sharp_{GD} = [I_1^\sharp, I_2^\sharp]_{SN}, \tag{6.16}$$

$$[S_1, S_2]^\flat_{SN} = [S_1^\flat, S_2^\flat]_{GD}. \tag{6.17}$$

Proof. Since (6.16) and (6.17) are equivalent, we only prove (6.17). Without loss of generality, we suppose

$$\begin{aligned} S_1 &= X_1 \bigwedge \cdots \bigwedge X_{k_1+1}, \\ S_2 &= Y_1 \bigwedge \cdots \bigwedge Y_{k_2+1}. \end{aligned}$$

By definition, we have

$$\begin{aligned} S_1^\flat &= \sum_i (-1)^{i-1} X_1 \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge \cdots \bigwedge X_{k_1+1} \bigotimes X_i, \\ S_2^\flat &= \sum_j (-1)^{j-1} Y_1 \bigwedge \cdots \bigwedge \widehat{Y_j} \bigwedge \cdots \bigwedge Y_{k_2+1} \bigotimes Y_j. \end{aligned}$$

Applying (6.3), we have

$$\begin{aligned} & [S_1^\flat, S_2^\flat]_{\mathrm{GD}} \\ = & \sum_{i,j} (-1)^{i+j} X_1 \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge \cdots \bigwedge X_{k_1+1} \bigwedge Y_1 \bigwedge \cdots \bigwedge \widehat{Y_j} \bigwedge \cdots \bigwedge Y_{k_2+1} \bigotimes [X_i, Y_j] \\ & + \sum_{i,j} (-1)^{i+j} X_1 \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge \cdots \bigwedge X_{k_1+1} \bigwedge ad_{X_i}(Y_1 \bigwedge \cdots \bigwedge \widehat{Y_j} \bigwedge \cdots \bigwedge Y_{k_2+1}) \bigotimes Y_j \\ & - \sum_{i,j} (-1)^{i+j} ad_{Y_j}(X_1 \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge \cdots \bigwedge X_{k_1+1}) \bigwedge Y_1 \bigwedge \cdots \bigwedge \widehat{Y_j} \bigwedge \cdots \bigwedge Y_{k_2+1} \bigotimes X_i \\ = & \sum_{i,j} (-1)^{i+j} X_1 \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge \cdots \bigwedge X_{k_1+1} \bigwedge Y_1 \bigwedge \cdots \bigwedge \widehat{Y_j} \bigwedge \cdots \bigwedge Y_{k_2+1} \bigotimes [X_i, Y_j] \\ & + \sum_{i,\, j > s} (-1)^{i+j+k_1+s-1} [X_i, Y_s] \bigwedge X_1 \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge \cdots \bigwedge X_{k_1+1} \bigwedge Y_1 \bigwedge \cdots \bigwedge \widehat{Y_s} \bigwedge \cdots \\ & \qquad \bigwedge \widehat{Y_j} \bigwedge \cdots \bigwedge Y_{k_2+1} \bigotimes Y_j \\ & + \sum_{i,\, j < s} (-1)^{i+j+k_1+s} [X_i, Y_s] \bigwedge X_1 \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge \cdots \bigwedge X_{k_1+1} \bigwedge Y_1 \bigwedge \cdots \bigwedge \widehat{Y_j} \bigwedge \cdots \\ & \qquad \bigwedge \widehat{Y_s} \bigwedge \cdots \bigwedge Y_{k_2+1} \bigotimes Y_j \\ & + \sum_{j,\, i > s} (-1)^{i+j+s-1} [X_s, Y_j] \bigwedge X_1 \bigwedge \cdots \bigwedge \widehat{X_s} \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge X_{k_1+1} \bigwedge Y_1 \bigwedge \\ & \qquad \cdots \bigwedge \widehat{Y_j} \bigwedge \cdots \bigwedge Y_{k_2+1} \bigotimes X_i \\ & + \sum_{j,\, i < s} (-1)^{i+j+s} [X_s, Y_j] \bigwedge X_1 \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge \cdots \bigwedge \widehat{X_s} \bigwedge X_{k_1+1} \bigwedge Y_1 \bigwedge \\ & \qquad \cdots \bigwedge \widehat{Y_j} \bigwedge \cdots \bigwedge Y_{k_2+1} \bigotimes X_i. \end{aligned}$$

By the definition of $\cdot^\flat$ and (2.10), we have

$$
\begin{aligned}
& [S_1, S_2]^\flat_{\mathrm{SN}} \\
= & \; (\sum_{i,j}(-1)^{i+j}[X_i, Y_j] \bigwedge X_1 \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge \cdots \bigwedge X_{k_1+1} \bigwedge Y_1 \bigwedge \cdots \bigwedge \widehat{Y_j} \bigwedge Y_{k_2+1})^\flat \\
= & \; \sum_{i,j}(-1)^{i+j} X_1 \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge \cdots \bigwedge X_{k_1+1} \bigwedge Y_1 \bigwedge \cdots \bigwedge \widehat{Y_j} \bigwedge \cdots \bigwedge Y_{k_2+1} \bigotimes [X_i, Y_j] \\
& + \sum_{j, s > i} (-1)^{i+j+s-1}[X_i, Y_j] \bigwedge X_1 \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge \cdots \bigwedge \widehat{X_s} \bigwedge \cdots \bigwedge X_{k_1+1} \bigwedge Y_1 \bigwedge \cdots \bigwedge \\
& \qquad \widehat{Y_j} \bigwedge \cdots \bigwedge Y_{k_2+1} \bigotimes X_s \\
& + \sum_{j, s < i} (-1)^{i+j+s}[X_i, Y_j] \bigwedge X_1 \bigwedge \cdots \bigwedge \widehat{X_s} \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge \cdots \bigwedge X_{k_1+1} \bigwedge Y_1 \bigwedge \cdots \bigwedge \\
& \qquad \widehat{Y_j} \bigwedge \cdots \bigwedge Y_{k_2+1} \bigotimes X_s \\
& + \sum_{i, s < j} (-1)^{i+j+k+s}[X_i, Y_j] \bigwedge X_1 \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge \cdots \bigwedge X_{k_1+1} \bigwedge Y_1 \bigwedge \cdots \bigwedge \widehat{Y_s} \bigwedge \cdots \\
& \qquad \widehat{Y_j} \bigwedge \cdots \bigwedge Y_{k_2+1} \bigotimes Y_s \\
& + \sum_{i, s > j} (-1)^{i+j+k+s-1}[X_i, Y_j] \bigwedge X_1 \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge \cdots \bigwedge X_{k_1+1} \bigwedge Y_1 \bigwedge \cdots \bigwedge \widehat{Y_j} \bigwedge \cdots \\
& \qquad \widehat{Y_s} \bigwedge \cdots \bigwedge Y_{k_2+1} \bigotimes Y_s.
\end{aligned}
$$

Exchanging the implicit indices i and s in the 2nd and 3rd terms, j and s in the 4th and 5th terms of the right hand side of this identity and comparing the result with what we already calculated for $[S_1^\flat, S_2^\flat]_{\mathrm{GD}}$, we have (6.17). □

This theorem embeds the Schouten-Nijenhuis algebra for the Lie algebra V into the Gelfand-Dorfman algebra. Therefore, it provides us an an expression of the Schouten-Nijenhuis bracket, which is defined on $V[1]$, in terms of elements in $Alt(V^*, V)$ through the embedding of $V[1]$ into $Alt(V^*, V)$. As remarked before, Gelfand and Dorfman were the first to give such kind of formula. Their result ([GD] and [D]) only involves the degree 1 elements. However, the Schouten-Nijenhuis algebra there is defined in a more general setting.

6.4 Drinfeld's Construction

At the very beginning of Poisson-Lie group and Lie bialgebra theory, Drinfeld ([Drl]) pointed out that Schouten-Nijenhuis algebras can be applied to describe a very important class of Poisson-Lie groups arising from the r-matrix formalism in the theory of integrable systems. In our terminology, his observation is the following:

Proposition 6.8 ([Drl]) *Let V be a Lie algebra and $r \in \bigwedge^2 V$. We have*

$$ad_X[r,r]_{SN} = 0 \qquad\qquad X \in V$$

if and only if the bracket on V^* determined by

$$[\psi_1, \psi_2] = -ad^*_{r^\flat(\psi_2)}\psi_1 + ad^*_{r^\flat(\psi_1)}\psi_2 \tag{6.18}$$

defines a Lie algebra.

A simple calculation will verify that Lie algebras V and V^*(with Lie bracket determined by (6.18)) constitute a Lie bialgebra ([Drl], see also [Lu]). It is usually called a coboundary Lie bialgebra.

In this section we want to show that when r in the above proposition is replaced by an arbitrary homogeneous element of the Schouten-Nijenhuis algebra certain condition exists so that we have a generalization of this result. The condition will be expressed in terms of the Nijenhuis-Richardson algebra $Alt(V^*, V^*)$ of the dual space of V.

For $I \in Alt^k(V^*, V)$, let

$$\begin{aligned} & L(I)(\psi_1, \cdots, \psi_{k+1}) \\ = \quad & \sum_{i=1}^{k+1}(-1)^{k+i-1} ad^*_{I(\psi_1,\cdots,\widehat{\psi_i},\cdots,\psi_{k+1})}\psi_i \\ = \quad & \sum_{\sigma\in sh(k-1,1)} (-1)^\sigma ad^*_{I(\psi_{\sigma(1)},\cdots,\psi_{\sigma(k-1)})}\psi_{\sigma(k)}. \end{aligned} \tag{6.19}$$

It is routine to check that $L(I) \in Alt^{k+1}(V^*, V^*)$. Therefore, we have a well-defined graded vector space homomorphism

$$L : Alt(V^*, V) \longrightarrow Alt(V^*, V^*)[1].$$

Note that under the isomorphism $Alt(V^*, V) = \bigwedge V \bigotimes V$, we have

$$L(T \bigotimes X) = T \bigwedge ad^*_X \tag{6.20}$$

for a simple tensor $T \bigotimes X$ in $\bigwedge V \bigotimes V$.

Theorem 6.9 *For $I_i \in Alt^{k_i}(V^*, V), i = 1, 2,$ we have*

$$L([I_1, I_2]_{GD}) = -[L(I_1), L(I_2)]_{NR}. \tag{6.21}$$

Proof. Without loss of generality, we suppose

$$I_i = T_i \bigotimes X_i, \qquad i = 1, 2.$$

By (6.20), we have

$$\begin{aligned} L([I_1, I_2]_{\mathrm{GD}}) &= L(T_1 \bigwedge T_2 \bigotimes [X_1, X_2] + T_1 \bigwedge ad_{X_1} T_2 \bigotimes X_2 - ad_{X_2} T_1 \bigwedge T_2 \bigotimes X_1) \\ &= T_1 \bigwedge T_2 \bigwedge ad^*_{[X_1, X_2]} + T_1 \bigwedge ad_{X_1} T_2 \bigwedge ad^*_{X_2} - ad_{X_2} T_1 \bigwedge T_2 \bigwedge ad^*_{X_1}. \end{aligned}$$

Hence,

$$\begin{aligned} & L([I_1, I_2]_{\mathrm{GD}})(\psi_1, \cdots, \psi_{k_1+k_2+1}) \\ = & \sum_{\sigma \in sh(k_1, k_2, 1)} (-1)^\sigma T_1(\psi_{\sigma^1}) T_2(\psi_{\sigma^2}) ad^*_{[X_1, X_2]} \psi_{\sigma(k_1+k_2+1)} \\ & - \sum_{\sigma \in sh(k_1, 1, k_2-1, 1)} (-1)^\sigma T_1(\psi_{\sigma^1}) T_2(ad^*_{X_1} \psi_{\sigma(k_1+1)}, \psi_{\sigma^3}) ad^*_{X_2} \psi_{\sigma(k_1+k_2+1)} \\ & + \sum_{\sigma \in sh(1, k_1-1, k_2, 1)} (-1)^\sigma T_1(ad^*_{X_2} \psi_{\sigma(1)}, \psi_{\sigma^2}) T_2(\psi_{\sigma^3}) ad^*_{X_1} \psi_{\sigma(k_1+k_2+1)}. \end{aligned} \tag{6.22}$$

Now, let us compute $[L(I_1), L(I_2)]_{\mathrm{NR}}$.

$$\begin{aligned} & L(I_2) L(I_1)(\psi_1, \cdots, \psi_{k_1+k_2+1}) \\ = & \sum_{\sigma \in sh(k_1+1, k_2)} (-1)^\sigma L(I_2)(L(I_1)(\psi_{\sigma^1}), \psi_{\sigma^2}) \\ = & \sum_{\sigma \in sh(k_1, 1, k_2)} (-1)^\sigma L(I_2)(T_1(\psi_{\sigma^1}) ad^*_{X_1} \psi_{\sigma(k_1+1)}, \psi_{\sigma^3}) \\ = & \sum_{\sigma \in sh(k_1, 1, k_2)} (-1)^\sigma (T_2 \bigwedge ad^*_{X_2})(T_1(\psi_{\sigma^1}) ad^*_{X_1} \psi_{\sigma(k_1+1)}, \psi_{\sigma^3}) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\sigma\in sh(k_1,1,k_2)} (-1)^\sigma(-1)^{k_2}T_1(\psi_{\sigma^1})T_2(\psi_{\sigma^2})ad^*_{X_2}ad^*_{X_1}\psi_{\sigma(k_1+1)} \\
&\quad + \sum_{\sigma\in sh(k_1,1,k_2)} (-1)^\sigma T_1(\psi_{\sigma^1})(\sum_{i=2}^{k_2+1}(-1)^{k_2+1+i}T_2(ad^*_{X_1}\psi_{\sigma(k_1+1)}, \\
&\qquad \psi_{\sigma(k_1+2)},\cdots,\widehat{\psi_{\sigma(k_1+i)}},\cdots, \\
&\qquad \psi_{\sigma(k_1+k_2+1)})ad^*_{X_2}\psi_{\sigma(k_1+i)}) \\
&= \sum_{\sigma\in sh(k_1,k_2,1)} (-1)^\sigma T_1(\psi_{\sigma^1})T_2(\psi_{\sigma^2})ad^*_{X_2}ad^*_{X_1}\psi_{\sigma(k_1+k_2+1)} \\
&\quad + \sum_{\sigma\in sh(k_1,1,k_2-1,1)} (-1)^\sigma T_1(\psi_{\sigma^1})T_2(ad^*_{X_1}\psi_{\sigma(k_1+1)},\psi_{\sigma^3})ad^*_{X_2}\psi_{\sigma(k_1+k_2+1)}
\end{aligned}
\tag{6.23}
$$

Similarly, we have

$$
\begin{aligned}
&L(I_1)L(I_2)(\psi_1,\cdots,\psi_{k_1+k_2+1}) \\
&= \sum_{\sigma\in sh(k_2,k_1,1)} (-1)^\sigma T_2(\psi_{\sigma^1})T_1(\psi_{\sigma^2})ad^*_{X_1}ad^*_{X_2}\psi_{\sigma(k_1+k_2+1)} \\
&\quad + \sum_{\sigma\in sh(k_2,1,k_1-1,1)} (-1)^\sigma T_2(\psi_{\sigma^1})T_1(ad^*_{X_2}\psi_{\sigma(k_2+1)},\psi_{\sigma^3})ad^*_{X_1}\psi_{\sigma(k_1+k_2+1)}
\end{aligned}
$$

Therefore, there holds

$$
\begin{aligned}
&-(-1)^{k_1k_2}L(I_1)L(I_2)(\psi_1,\cdots,\psi_{k_1+k_2+1}) \\
&= -\sum_{\sigma\in sh(k_1,k_2,1)} (-1)^\sigma T_1(\psi_{\sigma^1})T_2(\psi_{\sigma^2})ad^*_{X_1}ad^*_{X_2}\psi_{\sigma(k_1+k_2+1)} \\
&\quad - \sum_{\sigma\in sh(1,k_1-1,k_2,1)} (-1)^\sigma T_1(ad^*_{X_2}\psi_{\sigma(1)},\psi_{\sigma^2})T_2(\psi_{\sigma^3})ad^*_{X_1}\psi_{\sigma(k_1+k_2+1)}.
\end{aligned}
\tag{6.24}
$$

Adding (6.23) and (6.24), applying

$$ad^*_{X_1}ad^*_{X_2} - ad^*_{X_2}ad^*_{X_1} = ad^*_{[X_1,X_2]},$$

and comparing the result with (6.22), we get (6.21). ◻

We need one more result to attain a generalization of Proposition 6.8.

Theorem 6.10 *If $I \in cAlt^k(V^*,V)$, we have*

$$-< X, L(I)(\psi_1,\cdots,\psi_{k+1}) >=< ad_X I^\sharp, \psi_1 \bigwedge\cdots\bigwedge\psi_{k+1} > . \tag{6.25}$$

Proof. Without loss of generality, we assume

$$I^\sharp = X_1 \bigwedge \cdots \bigwedge X_{k+1},$$

then

$$\begin{aligned} I &= \sum_{i=1}^{k+1} (-1)^{i-1} X_1 \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge \cdots \bigwedge X_{k+1} \bigotimes X_i, \\ L(I) &= \sum_{i=1}^{k+1} (-1)^{i-1} X_1 \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge \cdots \bigwedge X_{k+1} \bigwedge ad^*_{X_i}. \end{aligned}$$

Hence,

$$\begin{aligned} & < X, L(I)(\psi_1, \cdots, \psi_{k+1}) > \\ = & < X, \sum_{i=1}^{k+1} (-1)^{i-1} (X_1 \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge \cdots \bigwedge X_{k+1} \bigotimes ad^*_{X_i})(\psi_1, \cdots, \psi_{k+1}) > \\ = & < X, \sum_{i=1}^{k+1} (X_1 \bigwedge \cdots \bigwedge ad^*_{X_i} \bigwedge \cdots \bigwedge X_{k+1})(\psi_1, \cdots, \psi_{k+1}) > \\ = & \sum_{i,j} (-1)^{i+j} < X, ad^*_{X_i}(\psi_j) >< X_1 \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge \cdots \bigwedge X_{k+1}, \psi_1 \bigwedge \cdots \bigwedge \widehat{\psi_j} \bigwedge \cdots \bigwedge \psi_{k+1} > \\ = & \sum_i (-\sum_j (-1)^{i+j} \psi_j([X, X_i]) \cdot < X_1 \bigwedge \cdots \bigwedge \widehat{X_i} \bigwedge \cdots \bigwedge X_{k+1}, \psi_1 \bigwedge \cdots \bigwedge \widehat{\psi_j} \bigwedge \cdots \bigwedge \psi_{k+1} >) \\ = & -\sum_{i=1}^{k+1} < X_1 \bigwedge \cdots \bigwedge [X, X_i] \bigwedge \cdots \bigwedge X_{k+1}, \psi_1 \bigwedge \cdots \bigwedge \psi_{k+1} > \\ = & - < ad_X(X_1 \bigwedge \cdots \bigwedge X_{k+1}), \psi_1 \bigwedge \cdots \bigwedge \psi_{k+1} > \\ = & - < ad_X I^\sharp, \psi_1 \bigwedge \cdots \bigwedge \psi_{k+1} > . \square \end{aligned}$$

Corollary 6.11 *If $I_i \in cAlt^{k_i}(V^*, V)$, we have*

$$\begin{aligned} & < X, [L(I_1), L(I_2)]_{NR} > \\ = & \; ad_X [I_1^\sharp, I_2^\sharp]_{SN} \end{aligned} \tag{6.26}$$

for any $X \in V$.

This corollary generalizes the formula [V3, (1.8)] (see also [K-SM] and [LX]) in the algebraic content. We expect it will be useful.

Proof. It follows from Theorem 6.7, Theorem 6.9 and Theorem 6.10. □

Corollary 6.12 *Let* $I \in cAlt^k(V^*, V)$*, then* $[L(I), L(I)]_{NR} = 0$ *if and only if* $ad_X[I^\sharp, I^\sharp]_{SN} = 0$ *for all* $X \in V$.

Proposition 6.8 is a special case of this corollary.

Chapter 7

The Generalized Nijenhuis-Richardson Algebra

In this chapter, we first generalize the Nijenhuis-Richardson algebra to the vector bundle case, then prove that this generalized Nijenhuis-Richardson algebra is isomorphic to two other interesting graded Lie algebras associated with a vector bundle. In the last section, through the introduction of $2n$-ary Lie algebroids, we give an example of the application of these isomorphisms.

7.1 Building the Algebra

We generalize the Nijenhuis-Richardson algebra from the vector space case to the vector bundle case in this section.

Our development will start from the semi-direct product structure $SP(V, W)$ associated with an infinite-dimensional vector space V and an infinite-dimensional Lie algebra W.

From now on, we will denote $[\ ]_{\mathrm{NR}}$ as $\{\ \}$ and $[\ ]_{\mathrm{LI}}$ as $[\ ]$ for simplicity.

Let A be a vector bundle on a smooth manifold M. We consider the vector space $V = \Gamma(A)$ of sections of A and the Lie algebra $W = \mathbf{X}(M)$ of vector fields over M.

Definition 7.1 *Let $(\varphi, \rho) \in Alt^{k+1}(\Gamma(A), \Gamma(A)) \oplus Alt^{k}(\Gamma(A), \mathbf{X}(M))$. (φ, ρ) is called a Lie-Rinehart pair of homogeneous degree k if we have the following:*

(7.1.a) *for any $f \in C^\infty(M)$, and $\xi_1, \cdots, \xi_k \in \Gamma(A)$,*

$$\rho(f\xi_1, \xi_2, \cdots, \xi_k) = f\rho(\xi_1, \xi_2, \cdots, \xi_k)$$

and

(7.1.b) *for any* $f \in C^\infty(M)$ *and* $\xi_1, \cdots, \xi_{k+1} \in \Gamma(A)$

$$\begin{aligned} & \varphi(f\xi_1, \xi_2, \cdots, \xi_{k+1}) \\ = \; & f\varphi(\xi_1, \xi_2, \cdots, \xi_{k+1}) - (-1)^k \rho(\xi_2, \xi_3, \cdots, \xi_{k+1}) f \xi_1. \end{aligned}$$

We denote $LR^k(A)$ for the space of all Lie-Rinehart pairs of homogeneous degree k and formulate the graded vector space

$$LR(A) = \oplus_{k \geq -1} LR^k(A) \tag{7.1}$$

$(LR^{-1}(A) = \Gamma(A))$. It is a subspace of the underlying graded vector space of $SP(\Gamma(A), \mathbf{X}(M))$.

Remark 7.2

(7.2.a) *Definition (7.1.a) is equivalent to* $\rho \in Alt^k_{C^\infty(M)}(\Gamma(A), \mathbf{X}(M))$, *i.e.,* ρ *is* $C^\infty(M)$-*linear. In other words,* ρ *is induced from a bundle map from* $\Lambda^k A$ *to* TM *([GHV]).*

(7.2.b) *By the alternating property, the condition (7.1.b) can be rewritten as*

$$\begin{aligned} & \varphi(\xi_1, \cdots, f\xi_i, \cdots, \xi_{k+1}) \\ = \; & f\varphi(\xi_1, \cdots, \xi_i, \cdots, \xi_{k+1}) + (-1)^{k+i} \rho(\xi, \cdots, \hat{\xi}_i, \cdots, \xi_{k+1}) f \cdot \xi_i \end{aligned}$$

for any $i = 1, 2, \cdots, k+1$.

The main result of this section is

Theorem 7.3 $LR(A)$ *is a subalgebra of the semidirect product* $SP(\Gamma(A), \mathbf{X}(M))$.

We will call this subalgebra $LR(A)$ the *generalized Nijenhuis-Richardson algebra* of the vector bundle A since when the vector bundle A degenerates to a vector space, this graded Lie algebra is just the Nijenhuis-Richardson algebra of the vector space.

Proof. Let $(\varphi_i, \rho_i) \in LR^{k_i}(A)$, $i = 1, 2$. We need to verify two identities,

$$\begin{aligned} & (\Im(\varphi_1)\rho_2 - (-1)^{k_1 k_2} \Im(\varphi_2)\rho_1 + [\rho_1, \rho_2])(f\xi_1, \xi_2, \cdots, \xi_{k_1+k_2}) \\ = \; & f(\Im(\varphi_1)\rho_2 - (-1)^{k_1 k_2} \Im(\varphi_2)\rho_1 + [\rho_1, \rho_2])(\xi_1, \xi_2, \cdots, \xi_{k_1+k_2}) \end{aligned} \tag{7.2}$$

and

$$
\begin{aligned}
&\{\varphi_1, \varphi_2\}(f\xi_1, \xi_2, \cdots, \xi_{k_1+k_2+1}) \\
= \;& f\{\varphi_1, \varphi_2\}(\xi_1, \xi_2, \cdots, \xi_{k_1+k_2+1}) \\
& -(-1)^{k_1+k_2}(\Im(\varphi_1)\rho_2 - (-1)^{k_1 k_2}\Im(\varphi_2)\rho_1 + [\rho_1, \rho_2])(\xi_2, \cdots, \xi_{k_1+k_2+1})f \cdot \xi_1.
\end{aligned}
\tag{7.3}
$$

Let us do (7.2) first.

By $(\varphi_i, \rho_i) \in LR^{k_i}(A)$, $i = 1, 2$, we have expansions

$$
\begin{aligned}
&\Im(\varphi_1)\rho_2(f\xi_1, \xi_2, \cdots, \xi_{k_1+k_2}) \\
= \;& f\Im(\varphi_1)\rho_2(\xi_1, \xi_2, \cdots, \xi_{k_1+k_2}) \\
& \overbrace{-(-1)^{k_1} \sum_{\sigma \in sh_1(k_1+1, k_2-1)} (-1)^{\sigma} \rho_1(\xi_{\sigma^1}) f \cdot \rho_2(\xi_1, \xi_{\sigma^2})}^{a}
\end{aligned}
$$

and

$$
\begin{aligned}
&\Im(\varphi_2)\rho_1(f\xi_1, \xi_2, \cdots, \xi_{k_1+k_2}) \\
= \;& f\Im(\varphi_2)\rho_1(\xi_1, \xi_2, \cdots, \xi_{k_1+k_2}) \\
& \overbrace{-(-1)^{k_2} \sum_{\sigma \in sh_1(k_2+1, k_1-1)} (-1)^{\sigma} \rho_2(\xi_{\sigma^1}) f \cdot \rho_1(\xi_1, \xi_{\sigma^2})}^{b}.
\end{aligned}
$$

$$
\begin{aligned}
&[f\rho_1, \rho_2](f\xi_1, \xi_2, \cdots, \xi_{k_1+k_2}) \\
= \;& f[\rho_1, \rho_2](\xi_1, \xi_2, \cdots, \xi_{k_1+k_2}) \\
& \overbrace{+ \sum_{\sigma \in sh_2(k_1, k_2)} (-1)^{\sigma} \rho_1(\xi_{\sigma^1}) f \cdot \rho_2(\xi_1, \xi_{\sigma^2})}^{a'} \\
& \overbrace{- \sum_{\sigma \in sh_1(k_1, k_2)} (-1)^{\sigma} \rho_2(\xi_{\sigma^2}) f \cdot \rho_1(\xi_1, \xi_{\sigma^1})}^{b'}.
\end{aligned}
$$

A careful combinatorial analysis shows

$$
\begin{aligned}
a + a' &= 0, \\
-(-1)^{k_1 k_2} b + b' &= 0.
\end{aligned}
$$

The identity (7.2) follows immediately.

As for (7.3), we can expand the left hand side into 12 terms,

$$\{\varphi_1,\varphi_2\}(f\xi_1,\xi_2,\cdots,\xi_{k_1+k_2+1})$$

$$= \overbrace{\sum_{\sigma\in sh_2(k_1+1,k_2)} (-1)^\sigma f\varphi_2(\varphi_1(\xi_{\sigma^1}),\xi_1,\xi_{\sigma^2})}^{a_1}$$

$$\overbrace{+(-1)^{k_2}\sum_{\sigma\in sh_2(k_1+1,k_2)} (-1)^\sigma \rho_2(\varphi_1(\xi_{\sigma^1}),\xi_{\sigma^2})f\cdot\xi_1}^{a_2}$$

$$\overbrace{-(-1)^{k_1k_2}\sum_{\sigma\in sh_2(k_2+1,k_1)} (-1)^\sigma f\varphi_1(\varphi_2(\xi_{\sigma^1}),\xi_1,\xi_{\sigma^2})}^{a_3}$$

$$\overbrace{-(-1)^{k_1}(-1)^{k_1k_2}\sum_{\sigma\in sh_2(k_2+1,k_1)} (-1)^\sigma \rho_1(\varphi_2(\xi_{\sigma^1}),\xi_{\sigma^2})f\cdot\xi_1}^{a_4}$$

$$+\overbrace{\sum_{\sigma\in sh_1(k_1+1,k_2)} (-1)^\sigma f\varphi_2(\varphi_1(\xi_1,\xi_{\sigma^1}),\xi_{\sigma^2})}^{a_5}$$

$$\overbrace{-(-1)^{k_2}\sum_{\sigma\in sh_1(k_1+1,k_2)} (-1)^\sigma \rho_2(\xi_{\sigma^2})f\cdot\varphi_1(\xi_1,\xi_{\sigma^1})}^{a_6}$$

$$\overbrace{-(-1)^{k_1}\sum_{\sigma\in sh_1(k_1+1,k_2)} (-1)^\sigma \rho_1(\xi_{\sigma^1})f\cdot\varphi_2(\xi_1,\xi_{\sigma^2})}^{a_7}$$

$$\overbrace{+(-1)^{k_1+k_2}\sum_{\sigma\in sh_1(k_1+1,k_2)} (-1)^\sigma \rho_2(\xi_{\sigma^2})\rho_1(\xi_{\sigma^1})f\cdot\xi_1}^{a_8}$$

$$\overbrace{-(-1)^{k_1k_2}\sum_{\sigma\in sh_1(k_2+1,k_1)}(-1)^{\sigma}f\varphi_1(\varphi_2(\xi_1,\xi_{\sigma^1}),\xi_{\sigma^2})}^{a_9}$$

$$\overbrace{+(-1)^{k_1}(-1)^{k_1k_2}\sum_{\sigma\in sh_1(k_2+1,k_1)}(-1)^{\sigma}\rho_1(\xi_{\sigma^2})f\cdot\varphi_2(\xi_1,\xi_{\sigma^1})}^{a_{10}}$$

$$\overbrace{+(-1)^{k_2}(-1)^{k_1k_2}\sum_{\sigma\in sh_1(k_2+1,k_1)}(-1)^{\sigma}\rho_2(\xi_{\sigma^1})f\cdot\varphi_1(\xi_1,\xi_{\sigma^2},)}^{a_{11}}$$

$$\overbrace{-(-1)^{k_1+k_2}(-1)^{k_1k_2}\sum_{\sigma\in sh_1(k_2+1,k_1)}(-1)^{\sigma}\rho_1(\xi_{\sigma^2})\rho_2(\xi_{\sigma^1})f\cdot\xi_1}^{a_{12}}$$

The following combinations can be checked through a careful count of signs:

$$\begin{aligned}
&a_1+a_3+a_5+a_9=f\{\varphi_1,\varphi_2\}(\xi_1,\xi_2,\cdots,\xi_{k_1+k_2+1})\\
&-(-1)^{k_1+k_2}a_2=\Im(\varphi_1)\rho_2(\xi_2,\cdots,\xi_{k_1+k_2+1})f\cdot\xi_1\\
&-(-1)^{k_1+k_2}a_4=-(-1)^{k_1k_2}\Im(\varphi_2)\rho_1(\xi_2,\cdots,\xi_{k_1+k_2+1})f\cdot\xi_1\\
&-(-1)^{k_1+k_2}(a_8+a_{12})=[\rho_1,\rho_2](\xi_2,\cdots,\xi_{k_1+k_2+1})f\cdot\xi_1\\
&a_6+a_{11}=0\\
&a_7+a_{10}=0
\end{aligned}$$

This completes the proof of (7.3), therefore that of Theorem 7.4.□

7.2 The Linear Schouten-Nijenhuis Algebra

In this section, we first embed the geometric Schouten-Nijenhuis algebra (i.e. the Schouten-Nijenhuis algebra over a manifold, see [MR]) into the Nijenhuis-Richardson algebra of the space of smooth functions on that manifold. Then, we point out that linear multiderivation fields on the vector bundle A^* constitute a subalgebra of the Schouten-Nijenhuis algebra of the manifold A^*, which we will call the linear Schouten-Nijenhuis on A^*. At the end, we prove the main theorem: the Nijenhuis-Richardson algebra on A is isomorphic to this linear Schouten-Nijenhuis algebra.

Let N be a smooth manifold and $V = C^\infty(N)$ denote the space of smooth functions on N.

Definition 7.4 *For $k > 0$, $S \in Alt^k(C^\infty(N), C^\infty(N))$ is called a k-derivation field on N if for any $f, g, f_2, \cdots, f_k \in C^\infty(N)$ there holds*

$$\begin{aligned} & S(fg, f_2, \cdots, f_k) \\ = \; & fS(g, f_2, \cdots, f_k) + gS(f, f_2, \cdots, f_k) \end{aligned} \tag{7.4}$$

i.e., $S(\cdot, f_2, \cdots, f_k)$ is a derivation of $C^\infty(N)$. We will denote $S^k(N)$ for the space of all k-derivation fields on N.

Remark 7.5 *The terminology k-derivation field is from [CKMV]. It is well-known that $S^1(N)$ can be identified with $\mathbf{X}(N) = \Gamma(TN)$ (see [FN1]). The same argument extends to the proof of the identification*

$$S^k(N) = \Gamma(\Lambda^k TN), \qquad k \geq 1. \tag{7.5}$$

Elements in $\Gamma(\Lambda^k TN)$ are usually called k-vector fields.

By the alternating property of S, (7.4) is equivalent to

$$\begin{aligned} & S(f_1, \cdots, f_{i-1}, fg, f_{i+1}, \cdots, f_k) \\ = \; & fS(f_1, \cdots, f_{i-1}, g, f_{i+1}, \cdots, f_k) \\ & + gS(f_1, \cdots, f_{i-1}, f, f_{i+1}, \cdots, f_k) \end{aligned} \tag{7.6}$$

for $f, g, f_1, \cdots, f_k \in C^\infty(N)$, $i = 1, 2, \cdots, k$.

We consider the graded vector space

$$S(N) = \bigoplus_{k \geq 0} S^k(N) \tag{7.7}$$

with $S^0(N) = C^\infty(N)$. Note that the Nijenhuis-Richardson algebra $Alt(C^\infty(N), C^\infty(N))[1]$ on $C^\infty(N)$ is well-defined. We have

Theorem 7.6 $S(N)[1]$ *is a subalgebra of the Nijenhuis-Richardson algebra* $Alt(C^\infty(N), C^\infty(N))[1]$.

Proof. Let $S_i \in S^{k_i+1}(N)$, $i = 1, 2$. We need to show $\{S_1, S_2\} \in S^{k_1+k_2+1}(N)$, i.e.,

$$\begin{aligned} & \{S_1, S_2\}(gh, f_2, \cdots, f_{k_1+k_2+1}) \\ = \; & g\{S_1, S_2\}(h, f_2, \cdots, f_{k_1+k_2+1}) \\ & + h\{S_1, S_2\}(g, f_2, \cdots, f_{k_1+k_2+1}) \end{aligned} \tag{7.8}$$

holds for any $h, g, f_2, \cdots, f_{k_1+k_2+1} \in C^\infty(N)$.

In fact,

$$\begin{aligned} & \{S_1, S_2\}(gh, f_2, \cdots, f_{k_1+k_2+1}) \\ = \; & \sum_{\sigma \in sh_1(k_1+1, k_2)} (-1)^\sigma S_2(S_1(gh, f_{\sigma^1}) \\ & + \sum_{\sigma \in sh_2(k_1+1, k_2)} (-1)^\sigma S_2(S_1(f_{\sigma^1}), gh, f_{\sigma^2}) \\ & -(-1)^{k_1 k_2} \sum_{\sigma \in sh_1(k_2+1, k_1)} (-1)^\sigma S_1(S_2(gh, f_{\sigma^1}), f_{\sigma^2}) \\ & -(-1)^{k_1 k_2} \sum_{\sigma \in sh_2(k_2+1, k_1)} (-1)^\sigma S_1(S_2(f_{\sigma^1}), gh, f_{\sigma^2}) \end{aligned} \tag{7.9}$$

The first term is

$$\overbrace{\sum_{\sigma\in sh_1(k_1+1,k_2)} (-1)^\sigma g S_2(S_1(h, f_{\sigma^1}), f_{\sigma^2})}^{b_1}$$

$$+ \overbrace{\sum_{\sigma\in sh_1(k_1+1,k_2)} (-1)^\sigma S_1(h, f_{\sigma^1}) \cdot S_2(g, f_{\sigma^2})}^{b_2}$$

$$+ \overbrace{\sum_{\sigma\in sh_1(k_1+1,k_2)} (-1)^\sigma h S_2(S_1(g, f_{\sigma^1}), f_{\sigma^2})}^{b_3}$$

$$+ \overbrace{\sum_{\sigma\in sh_1(k_1+1,k_2)} (-1)^\sigma S_1(g, f_{\sigma^1}) \cdot S_2(h, f_{\sigma^2})}^{b_4}.$$

The second term is

$$\overbrace{\sum_{\sigma\in sh_2(k_1+1,k_2)} (-1)^\sigma g S_2(S_1(f_{\sigma^1}), h, f_{\sigma^2})}^{b_5}$$

$$+ \overbrace{\sum_{\sigma\in sh_2(k_1+1,k_2)} (-1)^\sigma h S_2(S_1(f_{\sigma^1}), g, f_{\sigma^2})}^{b_6}$$

The third term is

$$\overbrace{-(-1)^{k_1k_2} \sum_{\sigma\in sh_1(k_2+1,k_1)} (-1)^\sigma g S_1(S_2(h, f_{\sigma^1}), f_{\sigma^2})}^{b_1'}$$

$$\overbrace{-(-1)^{k_1k_2} \sum_{\sigma\in sh_1(k_2+1,k_1)} (-1)^\sigma S_2(h, f_{\sigma^1}) \cdot S_1(g, f_{\sigma^2})}^{b_2'}$$

$$\overbrace{-(-1)^{k_1k_2} \sum_{\sigma\in sh_1(k_2+1,k_1)} (-1)^\sigma h S_1(S_2(g, f_{\sigma^1}), f_{\sigma^2})}^{b_3'}$$

$$\overbrace{-(-1)^{k_1k_2} \sum_{\sigma\in sh_1(k_2+1,k_1)} (-1)^\sigma S_2(g, f_{\sigma^1}) \cdot S_1(h, f_{\sigma^2})}^{b_4'}$$

And the fourth term is

$$\overbrace{-(-1)^{k_1k_2}\sum_{\sigma\in sh_2(k_2+1,k_1)}(-1)^{\sigma}gS_1(S_2(f_{\sigma^1}),h,f_{\sigma^2})}^{b'_5}$$

$$\overbrace{-(-1)^{k_1k_2}\sum_{\sigma\in sh_2(k_2+1,k_1)}(-1)^{\sigma}hS_2(S_1(f_{\sigma^1}),g,f_{\sigma^2})}^{b'_6}$$

We clearly have

$$\begin{aligned} b_1+b_5+b'_1+b'_5 &= g\{S_1,S_2\}(h,f_2,\cdots,f_{k_1+k_2+1}) \\ b_3+b_6+b'_3+b'_6 &= h\{S_1,S_2\}(g,f_2,\cdots,f_{k_1+k_2+1}) \end{aligned}$$

A combinatorial analysis gives us

$$\begin{aligned} b_2+b'_4 &= 0 \\ b_4+b'_2 &= 0 \end{aligned}$$

The proof of (7.8) is completed.□

Another proof of Theorem 7.6 is given in [CKMV]. The proof here seems to be more straightforward. Also in [CKMV] (see also [dWL]), the authors explain that with the identification (7.5), the graded Lie algebra $S(N)[1]$ is identical to the Schouten-Nijenhuis algebra of N. Therefore, we can reasonably call $S(N)[1]$ the Schouten-Nijenhuis algebra of N.

Now, we consider the special case $N=A^*$, the dual bundle of the vector bundle A.

It is clear that for any $\xi\in\Gamma(A)$, the function $l_\xi\in C^\infty(A^*)$,

$$l_\xi(\omega)=<\xi,\omega>,\quad \omega\in A^* \tag{7.10}$$

is fiber-linear. In fact, (7.10) identifies fibre-linear functions on A^* with $\Gamma(A)$.

If we denote $\pi: A^*\to M$ to be the projection of the vector bundle A^*, then the following two identities are obvious,

$$\pi^*(f_1f_2)=\pi^*(f_1)\pi^*(f_2) \tag{7.11}$$

$$\pi^*f\cdot l_\xi=l_{f\xi} \tag{7.12}$$

where π^* is the pull-back map, and $f, f_1, f_2 \in C^\infty(M)$, $\xi \in \Gamma(A)$.

We are interested in a special kind of multiderivation field on the manifold A^*.

Definition 7.7 *A k-derivation field $S \in S^k(A^*)$ is called linear if for any $\xi_1, \cdots, \xi_k \in \Gamma(A)$, there exists a unique $\xi \in \Gamma(A)$ such that*

$$S(l_{\xi_1}, \cdots, l_{\xi_k}) = l_\xi. \tag{7.13}$$

In other words, the value of a linear k-derivation field on linear functions is a linear function.

Denote $LS^k(A^*)$ the space of all linear k-derivation fields on A^*, $k \geq 0$ ($LS^0(A^*) = \{l_\xi : \xi \in \Gamma(A)\}$). We can define a graded vector space

$$LS(A^*) = \oplus_{k\geq 0} LS^k(A^*). \tag{7.14}$$

It is a subspace of $S(A^*)$. We can easily prove that

Proposition 7.8 *$LS(A^*)[1]$ is a subalgebra of $S(A^*)[1]$.*

We will call $LS(A^*)[1]$ the *linear Schouten-Nijenhuis algebra* of A^*.

Remark 7.9 *Let (x^a) be a local coordinate system on M and let $e_1, \cdots, e_n$ be a basis of local sections of A. We denote by (x^a, z_i) the corresponding coordinate system on A^*. Then*

$$l_{e_k} = z_k, \qquad k = 1, 2, \cdots, n.$$

Under identification (7.5), the linear multi-derivation field S becomes a linear polyvector field $\overline{S}$. It can be proved that $\overline{S} \in \Gamma(\Lambda^{k+1} TA^*)$ is a linear $(k+1)$-vector field if and only if, locally, it is of the form

$$\begin{aligned}\overline{S} &= \sum_{j, i_1 < \cdots < i_{k+1}} S^j_{i_1, \cdots, i_{k+1}} z_j \frac{\partial}{\partial z_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial z_{i_{k+1}}} \\ &\quad + \sum_{a, j_1 < \cdots < j_k} S^a_{j_1 \cdots j_k}(x) \frac{\partial}{\partial x^a} \wedge \frac{\partial}{\partial z_{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial z_{j_k}},\end{aligned}$$

where $S^j_{i_1 \cdots i_{k+1}}(x)$, $S^a_{j_1 \cdots j_k}(x) \in C^\infty(M)$. Proposition 7.8 is clear through this approach by applying the local formula for the usual Schouten-Nijenhuis algebra ([V1]).

The central goal of this section is to show that this linear Schouten-Nijenhuis algebra is isomorphic to the graded Lie algebra $LR(A)$.

We will construct a map $J : LS(A^*)[1] \longrightarrow LR(A)$.

Given a linear k-derivation field $S \in LS^{k+1}(A^*)$, for any $\xi_1, \cdots, \xi_{k+1} \in \Gamma(A)$, by Definition 7.7, we can define $\varphi(\xi_1, \cdots, \xi_{k+1}) \in \Gamma(A)$ through

$$S(l_{\xi_1}, \cdots, l_{\xi_{k+1}}) = l_{\varphi(\xi_1, \cdots, \xi_{k+1})}. \tag{7.15}$$

Then $\varphi \in Alt^{k+1}(\Gamma(A), \Gamma(A))$.

Now we show that S also uniquely defines a $\rho \in Alt^k_{C^\infty(M)}(\Gamma(A), \mathbf{X}(M))$.

Note that for any $f \in C^\infty(M)$ and $\xi_1, \cdots, \xi_{k+1} \in \Gamma(A)$, there holds

$$\begin{aligned} & S(l_{f\xi_1}, l_{\xi_2}, \cdots, l_{\xi_{k+1}}) \\ = \; & S(\pi^* f l_{\xi_1}, l_{\xi_2}, \cdots, l_{\xi_{k+1}}) \\ = \; & \pi^* f S(l_{\xi_1}, l_{\xi_2}, \cdots, l_{\xi_{k+1}}) + l_{\xi_1} S(\pi^* f, l_{\xi_2}, \cdots, l_{\xi_{k+1}}). \end{aligned}$$

In terms of φ, this can be rewritten as

$$\begin{aligned} & S(\pi^* f, l_{\xi_2}, \cdots, l_{\xi_{k+1}}) l_{\xi_1} \\ = \; & l_{\varphi(f\xi_1, \xi_2, \cdots, \xi_{k+1})} - l_{f\varphi(\xi_1, \xi_2, \cdots, \xi_{k+1})}. \end{aligned} \tag{7.16}$$

From this identity, we immediately have

$$S(\pi^* f, l_{\xi_2}, \cdots, l_{\xi_{k+1}}) \in \pi^* C^\infty(M).$$

Further, for any $g \in C^\infty(M)$, $\xi_2, \cdots, \xi_{k+1} \in \Gamma(A)$, there holds

$$\begin{aligned} & S(\pi^* f, l_{g\xi_2}, \cdots, l_{\xi_{k+1}}) \\ = \; & S(\pi^* f, \pi^* g l_{\xi_3}, \cdots, l_{\xi_{k+1}}) l_{\xi_2} \\ & + \pi^* g S(\pi^* f, l_{\xi_2}, \cdots, l_{\xi_{k+1}}). \end{aligned} \tag{7.17}$$

Since $S(\pi^* f, l_{g\xi_2}, \cdots, l_{\xi_{k+1}})$ and $\pi^a stgS(\pi^* f, l_{\xi_2}, \cdots, l_{\xi_{k+1}})$ are in $\pi^* C^\infty(M)$, we have

$$S(\pi^* f, \pi^* g, l_{\xi_3}, \cdots, l_{\xi_{k+1}}) l_{\xi_2} \in \pi^* C^\infty(M).$$

Because l_{ξ_2} is a fiber-linear function, we must have

$$S(\pi^* f, \pi^* g, l_{\xi_3}, \cdots, l_{\xi_{k+1}}) = 0.$$

Returning to (7.17), we have

$$\begin{aligned} & S(\pi^* f, l_{g\xi_2}, \cdots, l_{\xi_{k+1}}) \\ = \quad & \pi^* g S((\pi^* f, l_{\xi_2}, \cdots, l_{\xi_{k+1}}). \end{aligned} \tag{7.18}$$

Since S is a $(k+1)$-derivation field, we naturally have

$$\begin{aligned} & S(\pi^*(f_1 f_2), l_{\xi_2}, \cdots, l_{\xi_{k+1}}) \\ = \quad & \pi^* f_1 S(\pi^* f_2, l_{\xi_2}, \cdots, l_{\xi_{k+}}) + \pi^* f_2 S(\pi^* f_1, l_{\xi_2}, \cdots, l_{\xi_{k+1}}). \end{aligned} \tag{7.19}$$

Identities (7.18) and (7.19) hold for any $f, g \in C^\infty(M)$ and $\xi_2, \cdots, \xi_{k+1} \in \Gamma(A)$. Therefore, by the injectivity of π^*, we have a unique $\rho \in Alt^k_{C^\infty(M)}(\Gamma(A), \mathbf{X}(M))$ such that

$$\begin{aligned} & S(\pi^* f, l_{\xi_2}, \cdots, l_{\xi_{k+1}}) \\ = \quad & -(-1)^k \pi^*(\rho(\xi_2, \cdots, \xi_{k+1}) f). \end{aligned} \tag{7.20}$$

Further the above (φ, ρ) decided by S is in $LR^k(A)$. Actually, (7.16) can be rewritten in terms of ρ as

$$\begin{aligned} & l_{\varphi(f\xi_1, \xi_2, \cdots, \xi_{k+1})} \\ = \quad & l_{f\varphi(\xi_1, \xi_2, \cdots, \xi_{k+1})} - (-1)^k \rho(\xi_2, \cdots, \xi_{k+1}) f \cdot \xi_1, \end{aligned}$$

and this is nothing but

$$\varphi(f\xi_1, \xi_2, \cdots, \xi_{k+1}) = f\varphi(\xi_1, \xi_2, \cdots, \xi_{k+1}) - (-1)^k \rho(\xi_2, \cdots, \xi_{k+1}) f \cdot \xi_1.$$

We define the promised map J by

$$\begin{aligned} J : LS(A^*)[1] &\to LR(A) \\ J(S) &= (\varphi, \rho) \end{aligned}$$

(7.21)

The central result of this section is

Theorem 7.10 *J is a graded Lie algebra isomorphism.*

Proof. If $(\varphi, \rho) = 0$, then we have

$$S(l_{\xi_1}, \cdots, l_{\xi_{k+1}}) = 0,$$
$$S(\pi^* f, l_{\xi_2}, \cdots, l_{\xi_{k+1}}) = 0,$$
$$S(\pi^* f, \pi^* g, l_{\xi_3}, \cdots, l_{\xi_{k+1}}) = 0.$$

By the third identity, for any $h \in C^\infty(M)$, we have

$$\begin{aligned} 0 &= S(\pi^* f, \pi^* g, l_{h\xi_3}, \cdots, l_{\xi_{k+1}}) \\ &= S(\pi^* f, \pi^* g, \pi^* h, l_{\xi_4}, \cdots, l_{\xi_{k+1}}) l_{\xi_3} \\ &\quad + S(\pi^* f, \pi^* g, l_{\xi_3}, l_{\xi_4}, \cdots, l_{\xi_{k+1}}) \\ &= S(\pi^* f, \pi^* g, \pi^* h, l_{\xi_4}, \cdots, l_{\xi_{k+1}}), \end{aligned}$$

i.e.,

$$S(\pi^* f, \pi^* g, \pi^* h, l_{\xi_4}, \cdots, l_{\xi_{k+1}}) = 0.$$

Continuing with this approach, we can show

$$S(\pi^* f, \pi^* g, \cdots) = 0,$$

where "$\cdots$" represents elements of the form l_ξ for $\xi \in \Gamma(A)$, or $\pi^* h$ for $h \in C^\infty(M)$.

The value of $S \in S^{k+1}(A^*)$ is uniquely determined by its value on l_ξ, $\pi^* f$, $\xi \in \Gamma(A)$, $f \in C^\infty(M)$. Therefore, by (7.15) and (7.20), S must be 0 when $(\varphi, \rho) = 0$. That is, J is injective.

Given $(\varphi, \rho) \in LR^k(A)$, we define S through

$$\begin{aligned} S(l_{\xi_1}, l_{\xi_2}, \cdots, l_{\xi_{k+1}}) &= l_{\varphi(\xi_1, \cdots, \xi_{K+1})} \\ S(\pi^* f, l_{\xi_2}, \cdots, l_{\xi_{k+1}}) &= -(-1)^k \pi^*(\rho(\xi_2, \cdots, \xi_{k+1}) f) \\ S(\pi^* f, \pi^* g, \cdots) &= 0, \end{aligned}$$

where in the third identity, "$\cdots$" represents functions of form l_ξ for $\xi \in \Gamma(A)$, or $\pi^* f$ for $f \in C^\infty(M)$. It is easy to show that $S \in LS^{k+1}(A^*)$ and $J(S) = (\varphi, \rho)$. Hence, J is also surjective.

We are left to prove

$$J(S_1, S_2) = [J(S_1), J(S_2)],$$

i.e.,

$$J\{S_1, S_2\} = (\{\varphi_1, \varphi_2\}, \Im(\varphi_1)\rho_2 - (-1)^{k_1 k_2}\Im(\varphi_2)\rho_1 + [\rho_1, \rho_2]), \tag{7.22}$$

where $S_i \in LS^{k_i+1}(A^*)$, $J(S_i) = (\varphi_i, \rho_i)$, $i = 1, 2$ and h denotes the action of the Nijenhuis-Richardson algebra on the LI algebra as defined in the Subsection 3.1.3.

By definition of J, (7.22) is equivalent to

$$\begin{aligned} &\{S_1, S_2\}(l_{\xi_1}, \cdots, l_{\xi_{k_1+k_2+1}}) \\ = \ & l_{\{\varphi_1, \varphi_2\}(\xi_1, \cdots, \xi_{k_1+k_2+1})} \end{aligned} \tag{7.22$'$}$$

and

$$\begin{aligned} &\{S_1, S_2\}(\pi^* f, l_{\xi_2}, \cdots, l_{\xi_{k_1+k_2+1}}) \\ = \ & -(-1)^{k_1+k_2}\pi^*((\Im(\varphi_1)\rho_2 - (-)^{k_1 k_2}\Im(\varphi_2)\rho_1 + [\rho_1, \rho_2])(\xi_2, \cdots, \xi_{k_1+k_2+1})f). \end{aligned} \tag{7.22$''$}$$

The identity (7.22)$'$ follows directly from

$$\begin{aligned} &S_i(l_{\xi_1}, \cdots, l_{\xi_{k_i+1}}) \\ = \ & l_{\varphi_i(l_{\xi_1}, \cdots, l_{\xi_{k_i+1}})}, \qquad i = 1, 2. \end{aligned}$$

The proof of (7.22)$''$ is a long computation similar to that used in the proof of Theorem 7.3. We refrain from repeating it again.□

7.3 The Derivation Algebra

We prove in this section a second isomorphism theorem associated with the generalized Nijenhuis-Richardson $LR(A)$. Let us begin with some recollections about the algebra $\Gamma(\Lambda^* A^*)$ from Chapter II of [GHV].

The exterior algebra bundle of A^* is, by definition, the Whitney sum

$$\Lambda^* A^* = \oplus_{k \geq 0} \Lambda^k A^*,$$

where $\Lambda^0 A^* = M \times R$ is the rank 1 trivial bundle over M, and $\Lambda^k A^*$ is the k-th exterior power of A^*.

The following identifications of $C^\infty(M)$-modules will be used throughout this section:

$$\Gamma(\Lambda^k A^*) = \Lambda^k_{C^\infty(M)} \Gamma(A^*) = Alt^k_{C^\infty(M)}(\Gamma(A), C^\infty(M)). \tag{7.23}$$

With these identifications, we have

$$\begin{aligned} \Gamma(\Lambda^* A^*) &= \oplus_{k \geq 0} \Gamma(\Lambda^k A^*) = \oplus_{k \geq 0} (\Lambda^k_{C^\infty(M)} \Gamma(A^*)) \\ &= \oplus_{k \geq 0} (Alt^k_{C^\infty(M)}(\Gamma(A), C^\infty(M))). \end{aligned} \tag{7.24}$$

There is obviously an exterior algebra structure on $\Gamma(\Lambda^* A^*)$.

We will consider the derivation algebra $D\Gamma(\Lambda^* A^*)$ of this exterior algebra. Note that because of (7.23) any element of $D\Gamma(\Lambda^* A^*)$ is uniquely determined by its action on $C^\infty(M)$ and $\Gamma(A^*)$.

Given $(\varphi, \rho) \in LR^k(A)$, define $D(\varphi, \rho)$ on $f \in C^\infty(M)$ through

$$D(\varphi, \rho)(f)(\xi_1, \cdots, \xi_k) = \rho(\xi_1, \cdots, \xi_k) f \tag{7.25}$$

and for $\gamma \in \Gamma(A^*)$ through

$$\begin{aligned} & D(\varphi, \rho)\gamma(\xi_1, \cdots, \xi_{k+1}) \\ = \; & \gamma(\varphi(\xi_1, \cdots, \xi_{k+1})) + (-1)^k \sum_{i \geq 1} (-1)^{i-1} \rho(\xi_1, \cdots, \hat{\xi}_i, \cdots, \xi_{k+1}) \gamma(\xi_i). \end{aligned} \tag{7.26}$$

Since $\rho \in Alt^k_{C^\infty(M)}(\Gamma(A),\mathbf{X}(M))$, we have $D(\varphi,\rho)f \in \Gamma(\Lambda^k A^*)$ by (7.23). It is also clear from (7.25) that for $g \in C^\infty(M)$,

$$D(\varphi,\rho)(fg) = fD(\varphi,\rho)g + gD(\varphi,\rho)f.$$

Further, by Definition (7.1.b), we also have

$$\begin{aligned}
& D(\varphi,\rho)\gamma(g\xi_1,\xi_2,\cdots,\xi_{k+1}) \\
= \; & \gamma(g\varphi(\xi_1,\cdots,\xi_{k+1}) - (-1)^k\rho(\xi_2,\cdots,\xi_{k+1})g\cdot\xi_1) \\
& +(-1)^k\sum_{i\geq 1}(-1)^{i-1}g\rho(\xi_1,\cdots,\hat{\xi}_i,\cdots,\xi_{k+1})\gamma(\xi_i) \\
& +(-1)^k\rho(\xi_2,\cdots,\xi_{k+1})g\cdot\gamma(\xi_1) \\
= \; & gD(\varphi,\rho)\gamma(\xi_1,\cdots,\xi_{k+1}).
\end{aligned}$$

Hence, $D(\varphi,\rho)\gamma \in \Gamma(\Lambda^{k+1}A^*)$.

This argument shows that we can extend $D(\varphi,\rho)$ to a k-derivation of $\Gamma(\Lambda^* A^*)$. It is easy to verify that the action of this k-derivation $D(\varphi,\rho)$ on $\omega \in \Gamma(\Lambda^k A^*)$ is given by

$$\begin{aligned}
& D(\varphi,\rho)\omega(\xi_1,\cdots,\xi_{k+1}) \\
= \; & \sum_{\sigma\in sh(k+1,n-1)}(-1)^\sigma\omega(\varphi(\xi_{\sigma^1}),\xi_{\sigma^2}) \\
& + \sum_{\sigma\in(k,n)}(-1)^\sigma\rho(\xi_{\sigma^1})\omega(\xi_{\sigma^2}).
\end{aligned} \tag{7.27}$$

The construction of $D(\varphi,\rho)$ above determines a linear map

$$\begin{aligned}
& H : LR(A) \to D\Gamma(\Lambda^* A^*) \\
& H(\varphi,\rho) = D(\varphi,\rho).
\end{aligned} \tag{7.28}$$

We want to show that it is a graded Lie algebra isomorphism.

The injective property of H is clear from (7.25) and (7.26). Given any k-derivation Π of $\Gamma(\Lambda^* A^*)$, we define φ and ρ through

$$(\Pi f)(\xi_1,\cdots,\xi_k) = \rho(\xi_1,\cdots,\xi_k)f, \tag{7.25$'$}$$

$$\Pi\gamma(\varphi(\xi_1,\cdots,\xi_{k+1})) = \gamma(\xi_1,\cdots,\xi_{k+1}) - (-1)^k \sum_{i\geq 1}(-1)^{i-1}\rho(\xi_1,\cdots,\hat{\xi}_i,\cdots,\xi_{k+1})\gamma(\xi_i), \tag{7.26$'$}$$

where $f \in C^\infty(M)$ and $\gamma \in \Gamma(A^*)$. Since Π is a derivation of $\Gamma(\Lambda^* A^*)$, we have

$$\Pi(fg) = (\Pi f)g + f(\Pi g)$$

and

$$(\Pi f)(g\xi_1,\cdots,\xi_k) = g(\Pi f)(\xi_1,\cdots,\xi_k),$$

i.e., $\rho(\xi_1,\cdots,\xi_k) \in \mathbf{X}(M)$ and

$$\rho(g\xi_1,\cdots,\xi_k) = g\rho(\xi_1,\cdots,\xi_k).$$

Therefore, $\rho \in Alt^k_{C^\infty(M)}(\Gamma(A), C^\infty(M))$. The formula (7.26)$'$ satisfies Definition (7.1.b), hence, $(\varphi,\rho) \in LR^k(A)$. It is clear that $D(\varphi,\rho) = \Pi$. This proves the surjective property of H.

We are left to show that for $(\varphi_i,\rho_i) \in LR^{k_i}(A)$, $i = 1, 2$, there holds

$$H[(\varphi_1,\rho_1),(\varphi_2,\rho_2)] = [H(\varphi_1,\rho_1), H(\varphi_2,\rho_2)],$$

i.e.,

$$\begin{aligned} & D(\{\varphi_1,\varphi_2\}, \Im(\varphi_1)\rho_2 - (-1)^{k_1k_2}\Im(\varphi_2)\rho_1 + [\rho_1,\rho_2]) \\ = \;& [D(\varphi_1,\rho_1), D(\varphi_2,\rho_2)]. \end{aligned} \tag{7.29}$$

It is enough to check only that both sides of (7.29) act equally on $f \in C^\infty(M)$ and $\gamma \in \Gamma(A^*)$, respectively. In fact, by (7.25) and (7.27), we have

$$\begin{aligned} & D(\varphi_1,\rho_1)(D(\varphi_2,\rho_2)f)(\xi_1,\cdots,\xi_{k_1+k_2}) \\ = & \sum_{\sigma\in sh(k_1+1,k_2-1)} (-1)^\sigma (D(\varphi_2,\rho_2)f)(\varphi_1(\xi_{\sigma^1}),\xi_{\sigma^2}) \\ & + \sum_{\sigma\in sh(k_1,k_2)} (-1)^\sigma \rho_1(\xi_{\sigma^1})(D(\varphi_2,\rho_2)f)(\xi_{\sigma^2}) \\ = & \overbrace{\sum_{\sigma\in sh(k_1+1,k_2-1)} (-1)^\sigma \rho_2(\varphi_1(\xi_{\sigma^1}),\xi_{\sigma^2})f}^{a} \\ & + \overbrace{\sum_{\sigma\in sh(k_1,k_2)} (-1)^\sigma \rho_1(\xi_{\sigma^1})\rho_2(\xi_{\sigma^2})f}^{b}. \end{aligned}$$

Similarly,

$$D(\varphi_2,\rho_2)(D(\varphi_1,\rho_1)f)(\xi_1,\cdots,\xi_{k_1+k_2})$$

$$= \overbrace{\sum_{\sigma\in sh(k_2+1,k_1-1)} (-1)^\sigma \rho_1(\varphi_2(\xi_{\sigma^1}),\xi_{\sigma^2})f}^{a'}$$

$$+ \overbrace{\sum_{\sigma\in sh(k_2,k_1)} (-1)^\sigma \rho_2(\xi_{\sigma^1})\rho_1(\xi_{\sigma^2})f}^{b'}.$$

Note that

$$\begin{aligned} a &= \Im(\varphi_1)\rho_2(\xi_1,\cdots,\xi_{k_1+k_2})f, \\ a' &= \Im(\varphi_2)\rho_1(\xi_1,\cdots,\xi_{k_1+k_2})f, \\ b-(-1)^{k_1k_2}b' &= [\rho_1,\rho_2](\xi_1,\cdots,\xi_{k_1+k_2})f. \end{aligned}$$

Therefore,

$$\begin{aligned} &([D(\varphi_1,\rho_1),D(\varphi_2,\rho_2)]f)(\xi_1,\cdots,\xi_{k_1+k_2}) \\ = \; &(\Im(\varphi_1)\rho_2-(-1)^{k_1k_2}\Im(\varphi_2)\rho_1+[\rho_1,\rho_2])(\xi_1,\cdots,\xi_{k_1+k_2})f. \end{aligned}$$

This is exactly

$$\begin{aligned} &D(\{\varphi_1,\varphi_2\},\Im(\varphi_1)\rho_2-(-1)^{k_1k_2}\Im(\varphi_2)\rho_1+[\rho_1,\rho_2])f \\ = \; &[D(\varphi_1,\rho_1),D(\varphi_2,\rho_2)]f. \end{aligned}$$

Let $\gamma \in \Gamma(A^*)$.

$$
\begin{aligned}
& D(\varphi_1,\rho_1)(D(\varphi_2,\rho_2)\gamma)(\xi_1,\cdots,\xi_{k_1+k_2+1}) \\
= & \sum_{\sigma\in sh(k_1+1,k_2)} (-1)^{\sigma}(D(\varphi_2,\rho_2)\gamma)(\varphi_1(\xi_{\sigma^1}),\xi_{\sigma^2}) \\
& + \sum_{\sigma\in sh(k_1,k_2+1)} (-1)^{\sigma}\rho_1(\xi_{\sigma^1})(D(\varphi_2,\rho_2)\gamma)(\xi_{\sigma^2}) \\
= & \overbrace{\sum_{\sigma\in sh(k_1+1,k_2)} (-1)^{\sigma}\gamma(\varphi_2(\varphi_1(\xi_{\sigma^1}),\xi_{\sigma^2}))}^{a_1} \\
& \overbrace{(-1)^{k_2}\sum_{\sigma\in sh(k_1+1,k_2)} (-1)^{\sigma}\rho_2(\xi_{\sigma^2})\gamma(\varphi_1(\xi_{\sigma^1}))}^{a_2} \\
& \left.\begin{array}{l} +(-1)^{k_2}\displaystyle\sum_{\sigma\in sh(k_1+1,k_2)} (-1)^{\sigma}\sum_{k\geq 2}(-1)^{k-1}\rho_2(\varphi_1(\xi_{\sigma^1}),\xi_{\sigma(k_1+2)}, \\ \qquad\qquad\cdots,\hat{\xi}_{\sigma(k_1+k)},\cdots,\xi_{\sigma(k_1+k_2+1)})\gamma(\xi_{\sigma(k_1+k)}) \end{array}\right\}(a_3) \\
& + \overbrace{\sum_{\sigma\in sh(k_1,k_2+1)} (-1)^{\sigma}\rho_1(\xi_{\sigma^1})\gamma(\varphi_2(\xi_{\sigma^2}))}^{a_4} \\
& \left.\begin{array}{l} +(-1)^{k_2}\displaystyle\sum_{\sigma\in sh(k_1,k_2+1)} (-1)^{\sigma}\sum_{k\geq 1}(-1)^{k-1}\rho_1(\xi_{\sigma^1})\rho_2(\xi_{\sigma(k_1+1)}, \\ \qquad\qquad\cdots,\hat{\xi}_{\sigma(k_1+k)},\cdots,\xi_{\sigma(k_1+k_2+1)})\gamma(\xi_{\sigma(k_1+k)}). \end{array}\right\}(a_5)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
& D(\varphi_2,\rho_2)(D(\varphi_1,\rho_1)\gamma)(\xi_1,\cdots,\xi_{k_1+k_2+1}) \\
= & \overbrace{\sum_{\sigma\in sh(k_2+1,k_1)} (-1)^{\sigma}\gamma(\varphi_1(\varphi_2(\xi_{\sigma^1}),\xi_{\sigma^2}))}^{b_1} \\
& + \overbrace{(-1)^{k_1}\sum_{\sigma\in sh(k_2+1,k_1)} (-1)^{\sigma}\rho_1(\xi_{\sigma^2})\gamma(\varphi_2(\xi_{\sigma^1}))}^{b_2} \\
& \left.\begin{array}{l} +(-1)^{k_1}\displaystyle\sum_{\sigma\in sh(k_2+1,k_1)} (-1)^{\sigma}\sum_{k\geq 2}(-1)^{k-1}\rho_1(\varphi_2(\xi_{\sigma^1}),\xi_{\sigma(k_2+2)}, \\ \qquad\qquad\cdots,\hat{\xi}_{\sigma(k_2+k)},\cdots,\xi_{\sigma(k_1+k_2+1)})\gamma(\xi_{\sigma(k_2+k)}) \end{array}\right\}(b_3) \\
& + \overbrace{\sum_{\sigma\in sh(k_2,k_1+1)} (-1)^{\sigma}\rho_2(\xi_{\sigma^1})\gamma(\varphi_1(\xi_{\sigma^2}))}^{b_4} \\
& \left.\begin{array}{l} +(-1)^{k_1}\displaystyle\sum_{\sigma\in sh(k_2,k_1+1)} (-1)^{\sigma}\sum_{k\geq 1}(-1)^{k-1}\rho_2(\xi_{\sigma^1})\rho_1(\xi_{\sigma(k_2+1)}, \\ \qquad\qquad\cdots,\hat{\xi}_{\sigma(k_2+k)},\cdots,\xi_{\sigma(k_1+k_2+1)})\gamma(\xi_{\sigma(k_2+k)}). \end{array}\right\}(b_5)
\end{aligned}
$$

Note that

$$\begin{aligned}
a_1 - (-1)^{k_1 k_2} b_1 &= \gamma(\{\varphi_1, \varphi_2\}(\xi_1, \cdots, \xi_{k_1+k_2+1})), \\
a_2 - (-1)^{k_1 k_2} b_4 &= 0, \\
a_4 - (-1)^{k_1 k_2} b_2 &= 0.
\end{aligned}$$

In order to prove (7.29) for $\gamma \in \Gamma(A^*)$, we only have to show

$$\begin{aligned}
& a_3 + a_5 - (-1)^{k_1 k_2} b_3 - (-1)^{k_1 k_2} b_5 \\
= \; & (-1)^{k_1+k_2} \sum_{k \geq 1} (-1)^{k-1} (\Im(\varphi_1)\rho_2 - (-1)^{k_1 k_2} \Im(\varphi_2)\rho_1 + [\rho_1, \rho_2]) \\
& (\xi_1, \cdots, \hat{\xi}_k, \cdots, \xi_{k_1+k_2+1}) \gamma(\xi_k).
\end{aligned}$$

Checking by terms $\gamma(\xi_i)$, $i = 1, 2, \cdots, k_1 + k_2 + 1$, we have

$$\begin{aligned}
a_3 &= (-1)^{k_1+k_2} \sum_{k \geq 1} (-1)^{k-1} \Im(\varphi_1)\rho_2(\xi_1, \cdots, \hat{\xi}_k, \cdots, \xi_{k_1+k_2+1}) \gamma(\xi_k), \\
b_3 &= (-1)^{k_1+k_2} \sum_{k \geq 1} (-1)^{k-1} \Im(\varphi_2)\rho_1(\xi_1, \cdots, \hat{\xi}_k, \cdots, \xi_{k_1+k_2+1}) \gamma(\xi_k), \\
a_5 - (-1)^{k_1 k_2} b_5 &= (-1)^{k_1+k_2} \sum_{k \geq 1} (-1)^{k-1} [\rho_1, \rho_2](\xi_1, \cdots, \hat{\xi}_k, \cdots, \xi_{k_1+k_2+1}) \gamma(\xi_k).
\end{aligned}$$

This finishes the case $\gamma \in \Gamma(A^*)$ for the proof of (7.31), and hence completes the proof of the following theorem:

Theorem 7.11 *The map of (7.28) is a graded Lie algebra isomorphism.*

7.4 $2n$-ary Lie Algebroids

At almost the same time, Stasheff and his associates from homotopy theory([SL],[St]), Hanlon and Wachs from combinatorial algebra([HW]), Gnedbaye from cyclic cohomology([Gn]) and Azcárraga and Bueno from physics([dAPB]) came to be interested in a specific kind of higher order generalizations of Lie algebras. They brought up this object along different paths and with different motivations. However, a single identity, called "generalized Jacobi identity"(see (7.15.b) in [dAPB]), is the focus for all of them.

When this identity appeared, mathematicians who are familiar with Nijenhuis-Richardson algebras realized immediately that it can be expressed by Nijenhuis and Richardson's graded Lie bracket ([MV]) as a generalization of the usual Jacobi identity.

Definition 7.12 *Let V be a vector space. V is a $2n$-ary Lie algebra if there is a $2n$-ary bracket*

$$[\ \ ]: \underbrace{V \times \cdots \times V}_{2n} \to V$$

satisfying

(7.12.a) $[v_1, \cdots, v_{2n}] = (-1)^{\sigma}[v_{\sigma(1)}, \cdots, v_{\sigma(2n)}]$, *for* $\sigma \in \Sigma_{2n}$ *and* $v_1, \cdots, v_{2n} \in V$.

(7.12.b) $\sum_{\sigma \in sh(2n,2n-1)} (-1)^{\sigma}[[v_{\sigma(1)}, \cdots, v_{\sigma(2n)}], v_{\sigma(2n+1)}, \cdots, v_{\sigma(4n-1)}] = 0$, *for* $v_1, \cdots, v_{4n-1} \in V$.

Let us denote

$$\varphi(v_1, \cdots, v_{2n}) = -[v_1, \cdots, v_{2n}]. \tag{7.30}$$

Then the condition (7.12.a) is equivalent to $\varphi \in Alt^{2n}(V, V)$, while (7.12.b) is equivalent to the composition product

$$\varphi\varphi = 0 \tag{7.31}$$

which is again equivalent to

$$\{\varphi, \varphi\} = 0. \tag{7.32}$$

Therefore, $2n$-ary Lie algebra structures on a vector space V are defined by degree $2n-1$, bracket-square 0 elements of its Nijenhuis-Richardson algebra.

Remark 7.13 *While Definition 7.12 does make sense for odd-ary brackets, we restrict our attention to this even-ary case. The reason is that we want to use the equivalence* *between (7.12.b) and (7.31) so that we can make a neat exposition. In the odd-ary case, this equivalence does not exist because for φ of odd degree (7.32) is always true. For similar reasons, we will define higher order Lie algebroids and Poisson structures only for the even case in sequel.*

A $2n$-ary Lie algebra is obviously a generalization of a Lie algebra. We know a Lie algebroid is also a generalization of a Lie algebra. Considering these two generalizations, we have a natural question: What's the proper object on vector bundles which generalizes $2n$-ary Lie algebras on vector spaces? In this section we propose a definition of $2n$-ary Lie algebroid and give a brief discussion about the implications of the results obtained in the last several sections for this object.

Our proposal is

Definition 7.14 *Let $A \to M$ be a vector bundle. A $2n$-ary Lie algebroid structure on A consists of (1) a $2n$-ary Lie algebra on $\Gamma(A)$ and (2) a bundle map $\rho : \Lambda^{2n-1}A \to TM$ such that*

$$(i) \qquad \frac{1}{2}\sum_{\sigma\in sh(2n-1,2n-1)}(-1)^{\sigma}[\rho(\xi_{\sigma(1)},\cdots,\xi_{\sigma(2n-1)}),\rho(\xi_{\sigma(2n)},\cdots,\xi_{\sigma(4n-2)})]$$
$$= \sum_{\sigma\in sh(2n,2n-2)}(-1)^{\sigma}\rho([\xi_{\sigma(1)},\cdots,\xi_{\sigma(2n)}],\xi_{\sigma(2n+1)},\cdots,\xi_{\sigma(4n-2)})$$

and

(ii) $[f\xi_1,\xi_2,\cdots,\xi_{2n}] = f[\xi_1,\cdots,\xi_{2n}] - \rho(\xi_2,\cdots,\xi_{2n})f\cdot\xi_1$, *where* $f \in C^\infty(M)$ *and* $\xi_k \in \Gamma(A)$, $k = 1,2,\cdots,4n-2$.

We will call ρ the *anchor map* of this $2n$-ary Lie algebroid.

Remark 7.15

(7.15.a) *When $n = 1$, $2n$-ary Lie algebroids are just the usual Lie algebroids.*

(7.15.b) *When $M = pt$, a single point, $2n$-ary Lie algebroids degenerate to $2n$-ary Lie algebras.*

Many basic constructions for Lie algebroids ([Ma1]) and for $2n$-ary Lie algebras ([dAIPB]) can be carried out on this higher order Lie algebroid structure.

Let the $2n$-ary Lie algebra on $\Gamma(A)$ be defined by $\varphi \in Alt^{2n}(\Gamma(A),\Gamma(A))$ through

(7.30), then we can rewrite (i) and (ii) of Definition 7.14 as

$$
\begin{aligned}
(i)' \qquad & \sum_{\sigma\in sh(2n-1,2n-1)} (-1)^{\sigma}[\rho(\xi_{\sigma(1)},\cdots,\xi_{\sigma(2n-1)}),\rho(\xi_{\sigma(2n)},\cdots,\xi_{\sigma(4n-2)})]+ \\
& 2\sum_{\sigma\in sh(2n,2n-2)} (-1)^{\sigma}\rho(\varphi(\xi_{\sigma(1)},\cdots,\xi_{\sigma(2n)}),\xi_{\sigma(2n+1)},\cdots)=0, \\
(ii)' \qquad & \varphi(f\xi_1,\xi_2,\cdots,\xi_{2n}) = f\varphi(\xi_1,\cdots,\xi_{2n})+\rho(\xi_2,\cdots,\xi_{2n})f\cdot\xi_1.
\end{aligned}
$$

Since ρ is a bundle map, as a map in $Alt(\Gamma(A),\mathbf{X}(M))$, it is $C^\infty(M)$-linear. Hence, (ii)$'$ implies $(\varphi,\rho)\in LR^{2n-1}(A)$. We note that $\{\varphi,\varphi\}=0$ and (i)$'$ are nothing but

$$[(\varphi,\rho),(\varphi,\rho)]=0. \tag{7.33}$$

Therefore, the following proposition is proved.

Proposition 7.16 *A is a $2n$-ary Lie algebroid if and only if $(\varphi,\rho)\in LR^{2n-1}(A)$ and $[(\varphi,\rho),(\varphi,\rho)]=0$.*

If we want to specify (φ,ρ), we will write the $2n$-ary Lie algebroid as (A,φ,ρ).

This proposition in particular implies

Corollary 7.17 *Lie algebroid structures on a vector bundle A correspond bijectively to degree 1, bracket-square 0 elements of the graded Lie algebra $LR(A)$.*

Applying the isomorphism H between $LR(A)$ and $D\Gamma(\Lambda^*A^*)$, Proposition 7.16 also gives

Proposition 7.18 *$2n$-ary Lie algebroids on A correspond bijectively to $(2n-1)$-differentials of the graded Lie algebra $\Gamma(\Lambda^*A^*)$.*

The case $n=1$ of this proposition was proved in [K-SM] and [X].

Suppose the $2n$-ary Lie algebroid is defined by (φ,ρ), then the corresponding $(2n-1)$-differential is $D=D(\varphi,\rho)$. By (7.25) and (7.27), we can write the correspondence in this proposition explicitly as

$$Df(\xi_1,\cdots,\xi_{2n-1})=\rho(\xi_1,\cdots,\xi_{2n-1})f, \tag{7.34}$$

$$
\begin{aligned}
& D\omega(\xi_1,\cdots,\xi_{2n-1+k}) \\
= & \sum_{\sigma\in sh(2n-1,k)} (-1)^\sigma \rho(\xi_{\sigma(1)},\cdots,\xi_{\sigma(2n-1)})\omega(\xi_{\sigma(2n)},\cdots,\xi_{\sigma(2n-1+k)}) \\
& - \sum_{\sigma\in sh(2n,k-1)} (-1)^\sigma \omega([\xi_{\sigma(1)},\cdots,\xi_{\sigma(2n)}],\xi_{\sigma(2n+1)},\cdots),
\end{aligned}
\tag{7.35}
$$

where $\omega \in \Gamma(\Lambda^k A^*)$, $\xi_i \in \Gamma(A)$, $i = 1,\cdots,2n-1+k$.

The following concept is called a "generalized Poisson structure" in [dAPPB].

Definition 7.19 *Let N be a smooth manifold. A $2n$-ary Poisson structure on N is a $2n$-ary bracket*

$$\{\quad\} : \underbrace{C^\infty(N)\times\cdots\times C^\infty(N)}_{2n} \to C^\infty(N)$$

satisfying

(7.19.a) $\{f_1,\cdots,f_{2n}\} = (-1)^\sigma\{f_{\sigma(1)},\cdots,f_{\sigma(2n)}\}$.

(7.19.b) $\{gh,f_2,\cdots,f_{2n}\} = g\{h,f_2,\cdots,f_{2n}\} + h\{g,f_2,\cdots,f_{2n}\}$,

and

(7.19.c) $\sum_{\sigma\in sh(2n,2n-1)} (-1)^\sigma\{\{f_{\sigma(1)},\cdots,f_{\sigma(2n)}\},f_{\sigma(2n+1)},\cdots\} = 0$.

Let us denote

$$S(f_1,\cdots,f_{2n}) = -\{f_1,\cdots,f_{2n}\}. \tag{7.36}$$

Then, (7.19.a) and (7.19.b) above are equivalent to $S \in S^{2n}(N)$, while (7.19.c) is nothing but the composition product

$$SS = 0$$

which is further equivalent to

$$\{S,S\} = 0. \tag{7.37}$$

Therefore, $2n$-ary Poisson structures on a smooth manifold N are defined by degree $(2n-1)$, bracket-square 0 elements of the Schouten-Nijenhuis algebra of N.

A smooth manifold with a $2n$-ary Poisson structure on it will be called a $2n$-ary Poisson manifold. In the case discussed above, we will write (N, S) for the $2n$-ary Poisson manifold.

When it comes to vector bundles, such as A^*, a $2n$-ary Poisson structure is called linear if S is a linear $2n$-derivation field on the vector bundle.

By the isomorphism J between $LR(A)$ and $LS(A^*)[1]$, we have

Proposition 7.20 *$2n$-ary Lie algebroid structures on a vector bundle A are equivalent to linear $2n$-ary Poisson structures on its dual bundle A^*.*

When $n = 1$, this proposition gives the famous generalized Lie-Poisson construction of T. Courant and A. Weinstein ([C] and [CDW]).

The equivalence can be given explicitly. If the $2n$-ary Lie algebroid on A is defined by (φ, ρ), then the linear $2n$-ary Poisson structure on A^* is defined by $S = J^{-1}(\varphi, \rho)$. We have

$$\begin{cases} \{l_{\xi_1}, \cdots, l_{\xi_{2n}}\} = l_{[\xi_1, \cdots, \xi_{2n}]}, \\ \{\pi^* f, l_{\xi_2}, \cdots, l_{\xi_{2n}}\} = \pi^*(\rho(\xi_2, \cdots, \xi_{2n}) f), \\ \{\pi^* f, \pi^* g, \cdots\} = 0, \end{cases} \tag{7.38}$$

where $f, g \in C^\infty(M)$, $\xi_i \in \Gamma(A)$, $i = 1, 2, \cdots, 2n$, and in the third identity, "$\cdots$" represents elements of the form $\pi^* h$, $h \in C^\infty(M)$ or l_ξ, $\xi \in \Gamma(A)$.

The most prominent example of Lie algebroids is probably the cotangent bundle of a Poisson manifold ([V1]). A similar construction can also be carried out in our higher order case.

Proposition 7.21 *The cotangent bundle of a $2n$-ary Poisson manifold is a $2n$-ary Lie algebroid.*

We can prove this proposition through a technique called tangent lift ([GU] and [YI]). The complete lift is a homomorphism from the Schouten-Nijenhuis algebra $S(N)$ to the linear Schouten-Nijenhuis algebra $LS(TN)$ for a manifold N. Therefore, $2n$-ary Poisson structures on N are correspondent to linear $2n$-ary Poisson structures on TN. The proposition then follows from Proposition 7.20 with $A = T^*N$. Because the concepts and calculations needed in this proof are all included in [GU]. We will omit the details here.

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Appendix: Glossary of Important Symbols

We list important symbols used in this thesis. The number followed indicates the page where the symbol's definition or description is given.

$[\ ]_C$	24	i_X	17
$[\ ]_{FN}$	29	l_ξ	71
$[\ ]_{GD}$	46	$S(N)$	69
$[\ ]_{KP}$	31	$T(Q)$	40
$[\ ]'_{KP}$	42	$\mathbf{X}(M)$	4
$[\ ]_{KS}$	39	$\mathbf{X}(N)$	68
$[\ ]_{LI}$	18	$\Gamma(A)$	3
$[\ ]_{NR}$	16	$\Gamma(A^*)$	77
$[\ ]_{SN}$	15	ad_X	46
$[\ ]_{SP}$	21	ad_Q	44
$[\ ]_{FN(P)}$	34	ad^*_X	32
$\bigwedge V[1]$	15	$C^\infty(M)$	4
$\bigwedge V \bigotimes V$	48	$C^\infty(N)$	68
$\bigwedge V \bigotimes W$	47	$LR(A)$	64
$\bigwedge V^* \bigotimes V$	16	$LS(A^*)$	72
$\bigwedge V^* \bigotimes W$	18	$LS(A^*)[1]$	72
$\bigwedge TN$	15	$SP(V)$	22
$\bigwedge^* A^*$	77	$\Gamma(\bigwedge^* A^*)$	77
$.^\sharp$	55	$SP(V, W)$	21
$.^\flat$	54	$SP(\Gamma(A), \mathbf{X}(M))$	64
α	11,30	$D\Gamma(\bigwedge^* A^*)$	77
α'	43	$Alt(V, V)$	17
β	11,30	$Alt_C(V, V)$	30
β'	43	$Alt_{FN}(V, V)$	30
δ	22	$Alt_{KS}(V, V)$	39
θ	22	$Alt(V, V)[1]$	16
Θ	23	$Alt(V, W)$	17
τ	28	$Alt(V^*, V)$	48
$\hat{\tau}$	31	$Alt(V^*, V^*)$	58
ι	38	$Alt(V^*, V^*)[1]$	58
$\hat{\iota}$	43	$cAlt(V^*, V)$	50
$\mathfrak{I}$	11,19	$sh(n_1, \cdots, n_k)$	13
d	33	$sh_i(n_1, \cdots, n_k)$	13
D	22	J	75
H	78	L	58